Probability Theory

CS 70 Discussion 8B

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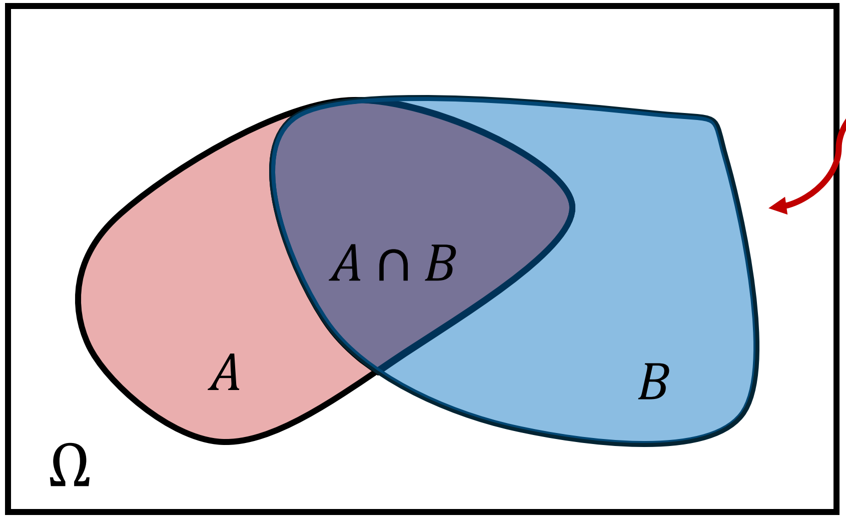
Note: These slides are unofficial course materials. Please use the notes as the only single source of truth.

Problem 1: Box of Marbles

Conditional probability:

$$\mathbb{P}[A|B] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]}$$

Probability of A given B occurs



New sample space

Problem 1: Box of Marbles

Trick 1:

- Law of total probability (useful when there are different cases $B_1, B_2, \ldots B_n$)

$$\mathbb{P}[A] = \sum_{i=1}^{n} \mathbb{P}[A \cap B_i]$$

Disjoint and covers Ω

$$= \sum_{i=1}^{n} \mathbb{P}[A|B_i]\mathbb{P}[B_i]$$

Probability of A if case B_i occurs × Probability B_i occurs

Problem 1: Box of Marbles

Trick 2:

- Bayes rule (useful when you know $\mathbb{P}[A|B]$ but want $\mathbb{P}[B|A]$)

$$\mathbb{P}[B|A] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[A]}$$

$$= \frac{\mathbb{P}[A|B]\mathbb{P}[B]}{\mathbb{P}[A]}$$

Often compute this using law of total probability

Problem 1: Box of Marbles

If you are done, solve the following problem

Q1: For (c), what if you looked at the color of the first ball? Does it changes the probability of second ball being blue? Why?

Q2: Let's generalize (c), suppose you sample k balls from Box 1 without looking at their color and put them aside, what is the probability that the next ball you drawn is blue?

Hint: Try the case $k = 2$, you should be able to "guess" the solution.
Making sense of the solution is the hard part…

Problem 1: Box of Marbles

Step 1: Define events

- A_i: We pick from box i
- B: The marble we picked is blue

Step 2: What probability are we trying to find?

$$\mathbb{P}[B]$$

Probability that the marble is blue

Intuition: the ball either comes from box 1 or box 2, so we can condition on this.

Problem 1: Box of Marbles

By law of total probability:

$$\mathbb{P}[B] = \mathbb{P}[B|A_1]\mathbb{P}[A_1] + \mathbb{P}[B|A_2]\mathbb{P}[A_2]$$

Probability that the marble is blue, if sample from box 1 × Probability we sample from box 1

- $\mathbb{P}[B|A_1] = 0.1, \mathbb{P}[B|A_2] = 0.5$
- $\mathbb{P}[A_1] = 0.5, \mathbb{P}[A_2] = 0.5$

$$= 0.1 \cdot 0.5 + 0.5 \cdot 0.5 = 0.3$$

Problem 1: Box of Marbles

Step 2: We now want to to find:

$$\mathbb{P}[A_1|B]$$

Probability that we sampled from the first bin given the ball sampled is blue.

Classic problem: We know $\mathbb{P}[B|A_1]$, but want $\mathbb{P}[A_1|B]$.

$$\mathbb{P}[A_1|B] = \frac{\mathbb{P}[A_1 \cap B]}{\mathbb{P}[B]} = \frac{\mathbb{P}[B|A_1]\mathbb{P}[A_1]}{\mathbb{P}[B]}$$

$$= \frac{0.1 \cdot 0.5}{0.3} = \frac{1}{6}$$

$$\mathbb{P}[B] = \mathbb{P}[B|A_1]\mathbb{P}[A_1] + \mathbb{P}[B|A_2]\mathbb{P}[A_2]$$

Problem 1: Box of Marbles

Step 1: Define events

- B_1, R_1: The first marble is blue/red
- B_2, R_2: The second marble is blue/red

Step 2: What probability are we trying to find?

$$\mathbb{P}[B_2]$$

Probability that the second marble is blue

This probability depends on whether the first ball is red or blue!

$$\mathbb{P}[B_2] = \mathbb{P}[\mathrm{B}_2|\mathrm{B}_1]\mathbb{P}[B_1] + \mathbb{P}[B_2|R_1]\mathbb{P}[R_1]$$

$$= \frac{99}{999} \cdot 0.1 + \frac{100}{999} \cdot 0.9 = 0.1$$

Suprisingly clean?

Problem 1: Box of Marbles

Now suppose we pick two balls without looking at their color, what is the probability that the next ball is blue?

$$\mathbb{P}[B_3] = \boxed{\mathbb{P}[\mathrm{B}_3|\mathrm{B}_2,\mathrm{B}_1]\mathbb{P}[B_2,B_1]} + \mathbb{P}[B_3|B_2,R_1]\mathbb{P}[B_2,R_1]$$

$$+ \mathbb{P}[\mathrm{B}_3|\mathrm{R}_2,\mathrm{B}_1]\mathbb{P}[R_2,B_1] + \mathbb{P}[B_3|R_2,R_1]\mathbb{P}[R_2,R_1]$$

$$= \boxed{\frac{98}{998}\cdot\frac{99}{999}\cdot\frac{100}{1000}} + \frac{99}{998}\cdot\frac{100}{999}\cdot\frac{900}{1000}$$

$$+ \frac{99}{998}\cdot\frac{900}{999}\cdot\frac{100}{1000} + \frac{100}{998}\cdot\frac{899}{999}\cdot\frac{900}{1000}$$

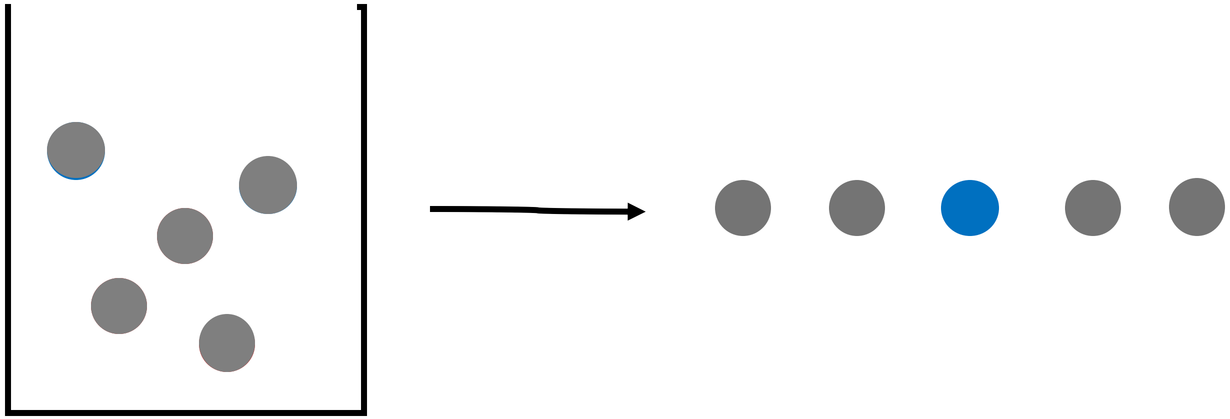
$\mathbb{P}[\mathrm{B}_3|\mathrm{B}_2,\mathrm{B}_1]\mathbb{P}[B_2|B_1]\mathbb{P}[B_1]$

Problem 1: Box of Marbles

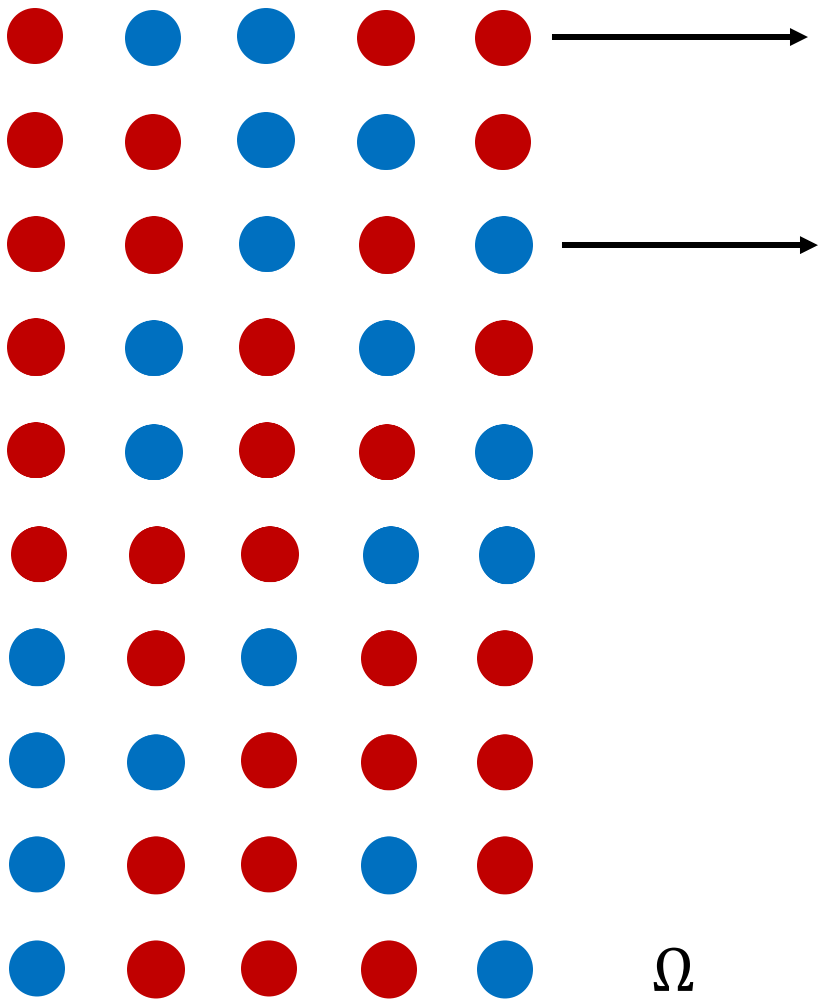
Simple case: Suppose there are 5 balls of which 2 is blue, 3 is red

Sample 2 balls without looking at their color, what is the probability next ball is blue?

Equivalent experiment: sample all 5 balls one by one but only reveal the second ball



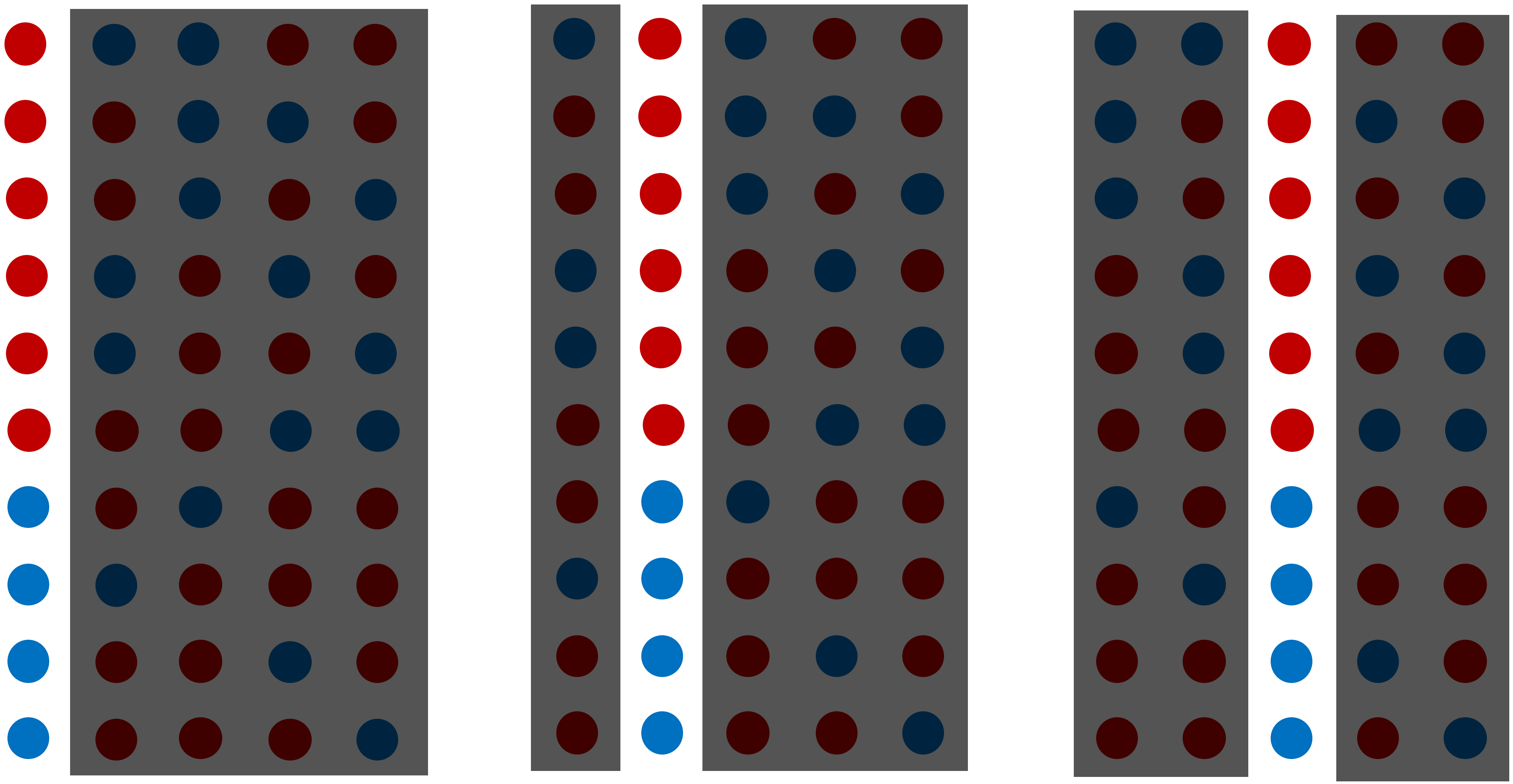
Problem 1: Box of Marbles



$$\frac{3}{5}\cdot\frac{2}{4}\cdot\frac{1}{3}\cdot\frac{2}{2}\cdot\frac{1}{1}=\frac{1}{10}$$

$$\frac{3}{5}\cdot\frac{2}{4}\cdot\frac{2}{3}\cdot\frac{1}{2}\cdot\frac{1}{1}=\frac{1}{10}$$

Problem 1: Box of Marbles



Problem 1: Box of Marbles

Formally, let:

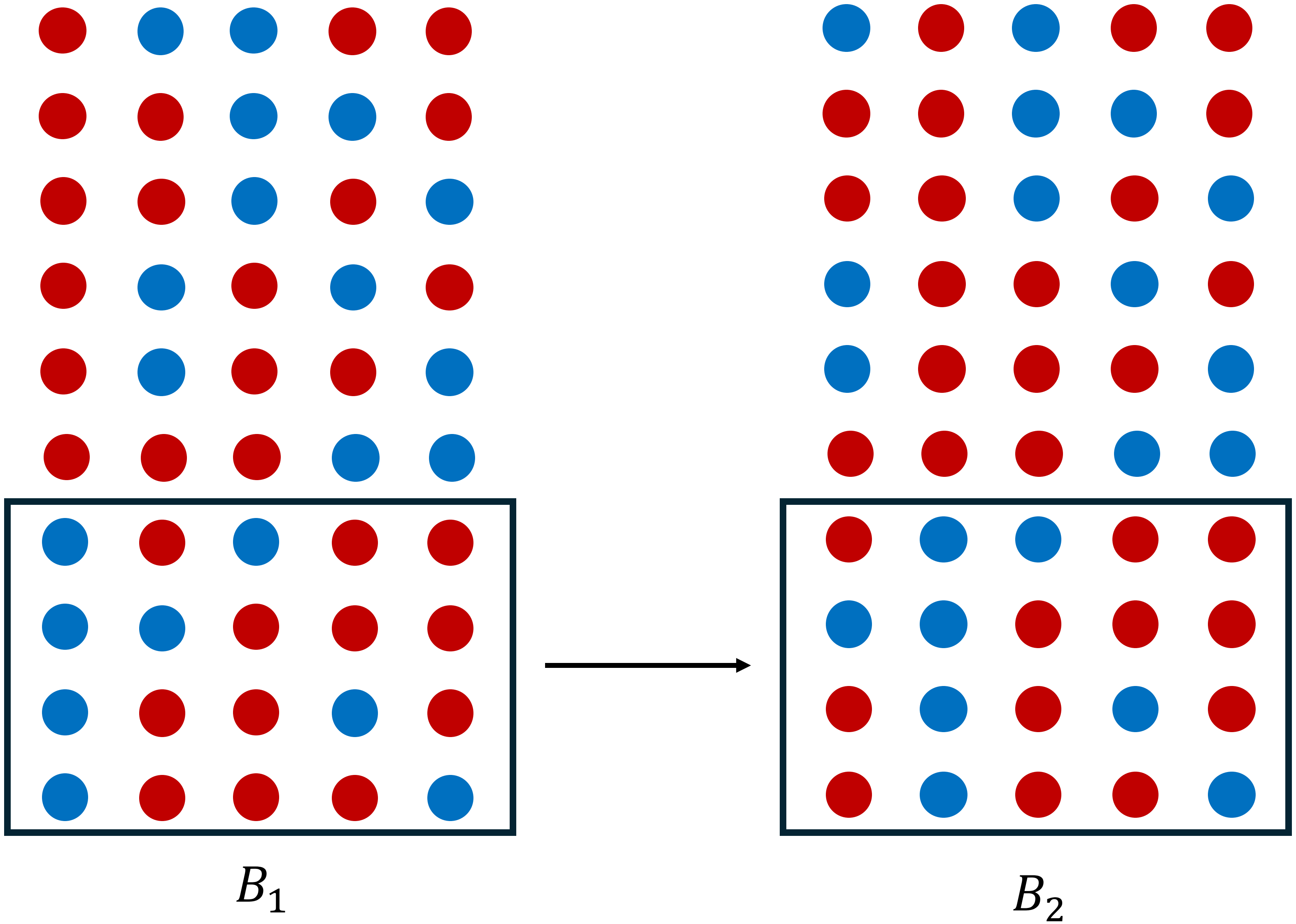
- B_1: first ball is blue
- B_2: second ball is blue, without revealing ball 1
- B_3: third ball is blue, without revealing ball 1 and 2

$$\mathbb{P}[B_i] = \frac{|B_i|}{|\Omega|}$$

Claim: $|B_1| = |B_2|$

Proof: There is a bijection $\mathrm{f}: B_1 \to B_2$ by swapping the first and second column

Problem 1: Box of Marbles



Problem 2: Duelling Meteorologists

Step 1: Define events

- S: It actually snows
- T: Tom predicts snow

Step 2: What probability are we trying to find?

$$\mathbb{P}[S|T]$$

From the given problem, we know

- $\mathbb{P}[T|S] = 0.7$
- $\mathbb{P}[T^C|S^C] = 0.95$
- $\mathbb{P}[S] = 0.1$

Problem 2: Duelling Meteorologists

- $\mathbb{P}[T|S] = 0.7$
- $\mathbb{P}[T^C|S^C] = 0.95$
- $\mathbb{P}[S] = 0.1$

$$\mathbb{P}[S|T] = \frac{\mathbb{P}[T|S]\mathbb{P}[S]}{\mathbb{P}[T]}$$

$$= \frac{\mathbb{P}[T|S]\mathbb{P}[S]}{\mathbb{P}[T|S]\mathbb{P}[S] + \mathbb{P}[T|S^C]\mathbb{P}[S^C]}$$

$$= \frac{0.7 \cdot 0.1}{0.7 \cdot 0.1 + 0.05 \cdot 0.9} = \frac{14}{23}$$

Problem 2: Duelling Meteorologists

- $\mathbb{P}[T|S] = 0.7$
- $\mathbb{P}[T^C|S^C] = 0.95$
- $\mathbb{P}[S] = 0.1$

$$\mathbb{P}[S \cap T] + \mathbb{P}[S^C \cap T^C] = \mathbb{P}[T|S]\mathbb{P}[S] + \mathbb{P}[T^C|S^C]\mathbb{P}[S^C]$$

$$= 0.1 \cdot 0.7 + 0.95 \cdot 0.9 = \frac{37}{40}$$

Problem 3: Independence

Two events A and B are independent if

$$\mathbb{P}[A \cap B] = \mathbb{P}[A]\mathbb{P}[B]$$

A more reasonable definition:

$$\mathbb{P}[A|B] = \mathbb{P}[A]$$

or

$$\mathbb{P}[B|A] = \mathbb{P}[B]$$

i.e. Whether B occurs or not does not change the probability of A