Probability Theory

CS 70 Discussion 8A

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Note: These slides are unofficial course materials. Please use the notes as the only single source of truth.

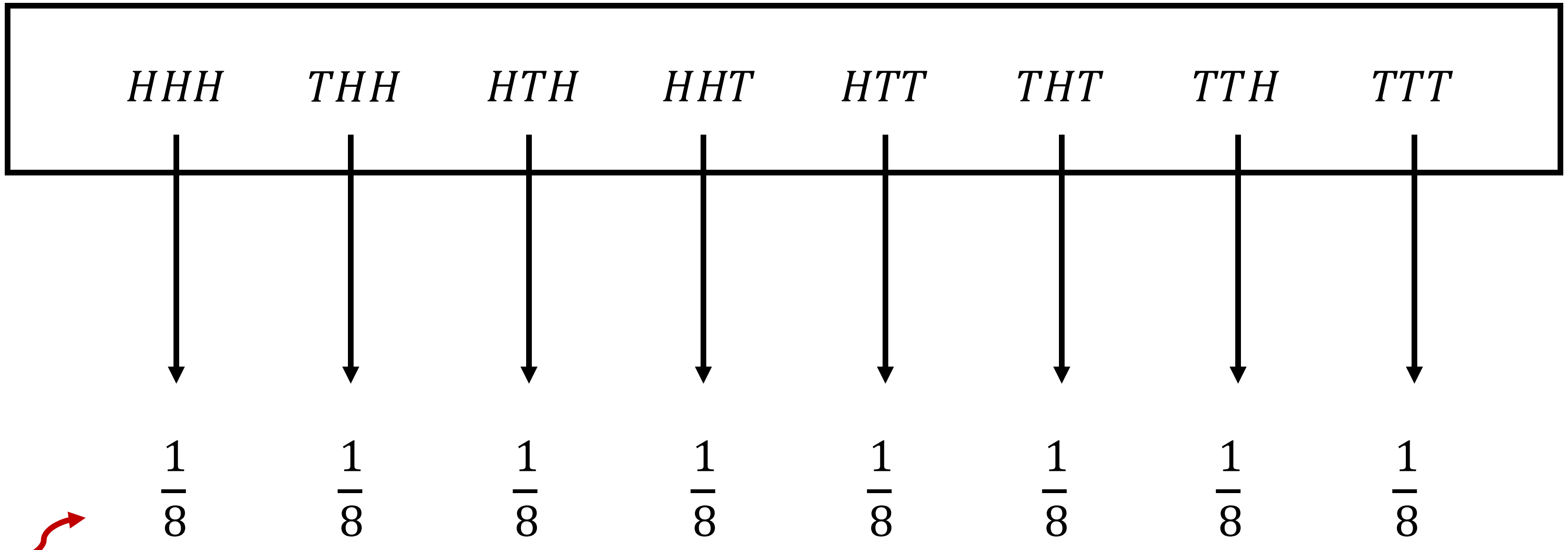
Problem 2: Flippin' Coins

Some additional questions to consider for problem 2:

Q1: What if the coin is biased? Say $\mathbb{P}[H] = \frac{1}{3}, \mathbb{P}[T] = \frac{2}{3}$

Q2: Now extend this to tossing n unbiased coins, what is the probability that you get k heads?

Problem 2: Flippin' Coins



Ω

Sample space

$\mathbb{P}$

Probability Measure

Interpretation 1:

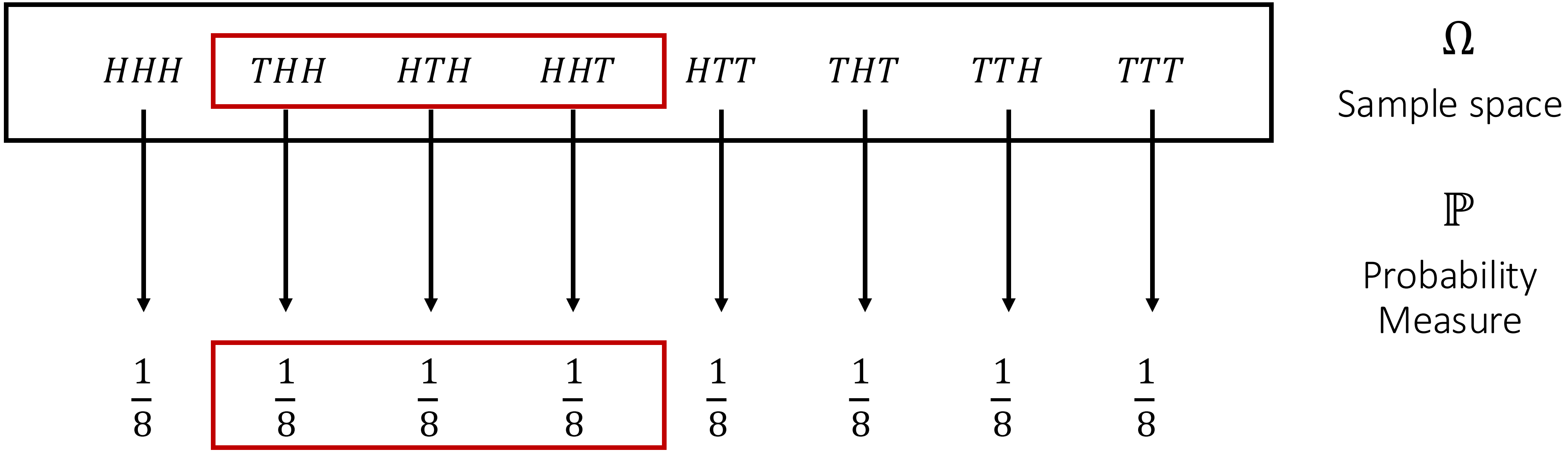
Uniform probability, so

$$\frac{1}{|\Omega|}$$

Interpretation 2:

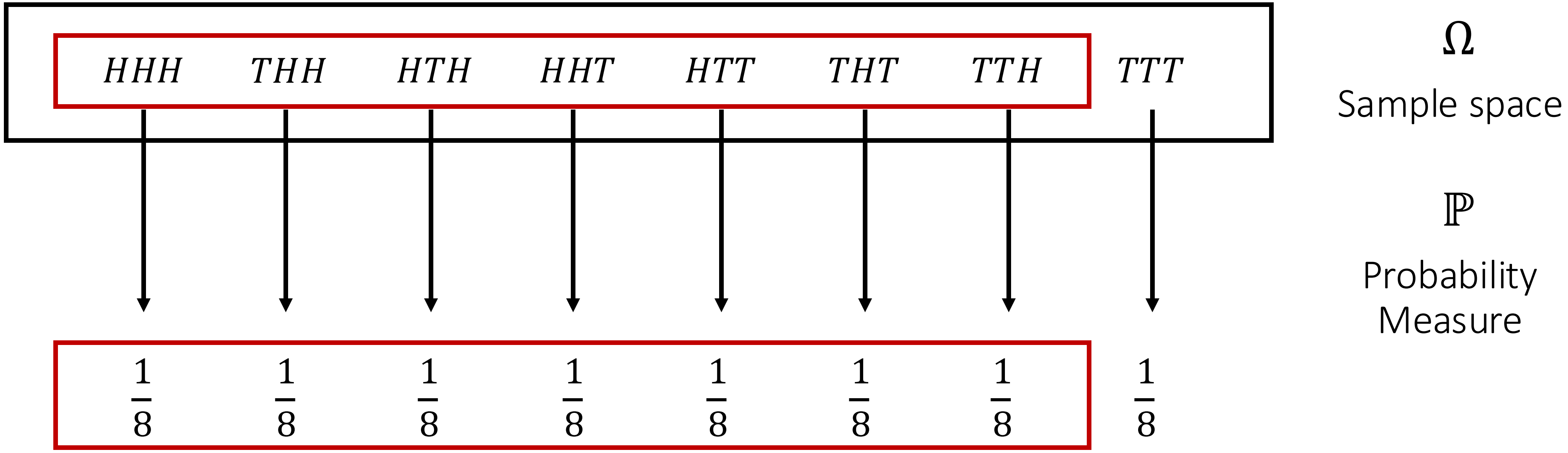
$$\begin{array}{ccccccc} T & & T & & T & & \\ \frac{1}{2} & \cdot & \frac{1}{2} & \cdot & \frac{1}{2} & = & \frac{1}{8} \end{array}$$

Problem 2: Flippin' Coins



$$\mathbb{P}[two\ heads] = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}$$

Problem 2: Flippin' Coins

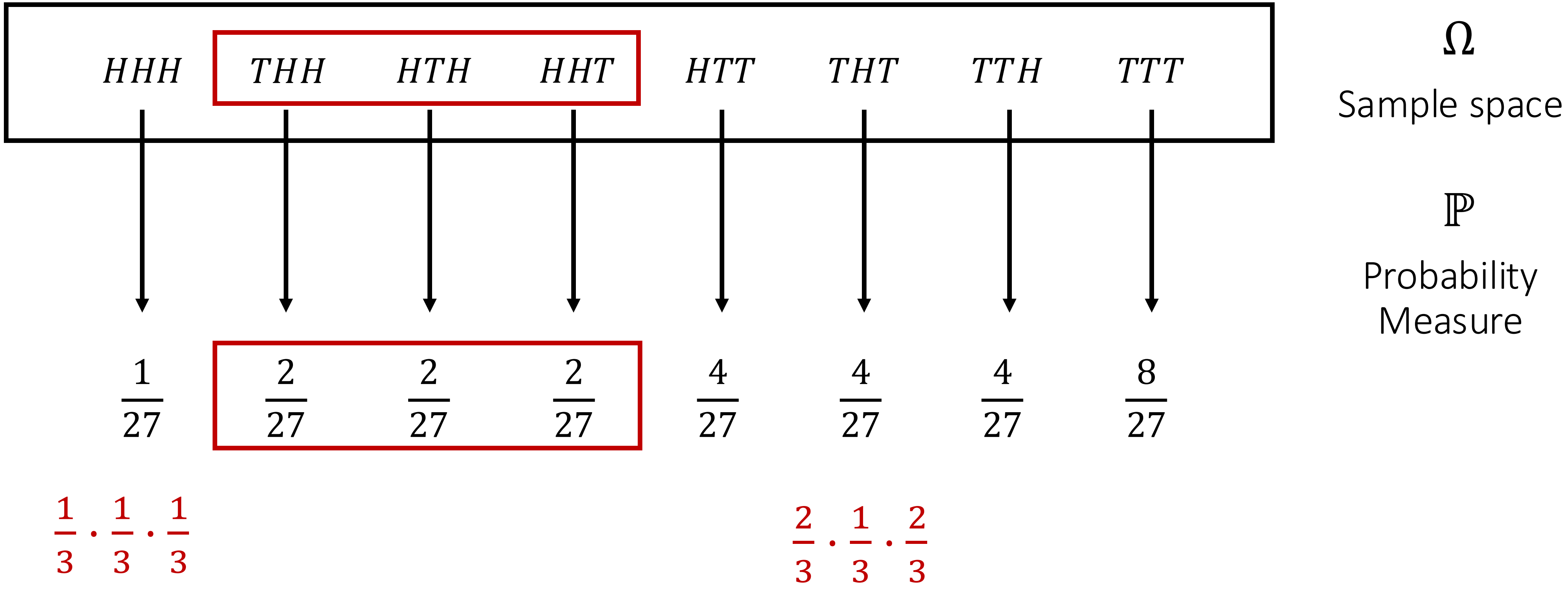


$$\mathbb{P}[\mathit{at\ least\ one\ head}] = 1 - \mathbb{P}[\mathit{zero\ head}]$$

$$= 1 - \frac{1}{8} = \frac{7}{8}$$

Problem 2: Flippin' Coins

What if $\mathbb{P}[H] = \frac{1}{3}, \mathbb{P}[T] = \frac{2}{3}$?



Problem 2: Flippin' Coins

Solving probability problems, first identify (ALWAYS!)

- Sample space Ω
- Probability measure $\mathbb{P}$

To compute probability of an event A:

Sum of probability weights in A

$$\mathbb{P}[A] = \sum_{w \in A} \mathbb{P}[w]$$

Usually a counting problem

If working with uniform probability:

$$= \sum_{w \in A} \frac{1}{|\Omega|} = \frac{|A|}{|\Omega|}$$

Problem 2: Flippin' Coins

Toss n unbiased coins, what is the probability of getting k heads?

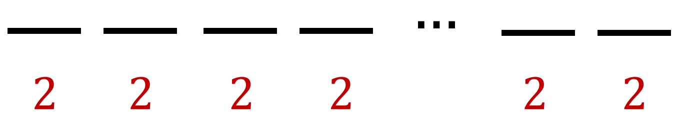
- Sample space Ω

Length n combinations of H and T

$$\Omega = \{\mathrm{HHH..H}, \mathrm{THH\ldots H}, \ldots\}$$

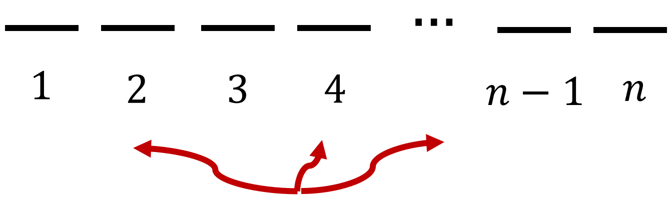
- Probability measure $\mathbb{P}$

$$\mathbb{P}[\mathrm{w}] = \frac{1}{|\Omega|} = \frac{1}{2^n}$$



What is the probability of getting k heads? A: Set of all length n sequence with k heads

$$\mathbb{P}[\mathrm{A}] = \frac{|A|}{|\Omega|} = \frac{\binom{n}{k}}{2^n}$$



From n indices choose k indices to put H

Problem 3: Sampling

An urn contains n balls, of which one is “special”, suppose you sample k balls at once, what is the probability you get the “special” ball?

Problem 3: Sampling

What is the sample space and probability function?

Second ball

First ball

$$
\begin{array}{ccccc}
(1,1) & (1,2) & (1,3) & \cdots & (1,n) \\
(2,1) & (2,2) & (2,3) & \cdots & (2,n) \\
(3,1) & (3,2) & (3,3) & \cdots & (2,n) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
(n,1) & (n,2) & (n,3) & \cdots & (n,n)
\end{array}
$$

$\downarrow$

$$\frac{1}{n^2}$$

$(1,1)$

$$\frac{1}{n} \cdot \frac{1}{n} = \frac{1}{n^2}$$

Problem 3: Sampling

$$
\begin{matrix}
(1,1) & (1,2) & (1,3) & \cdots & (1,n) \\
(2,1) & (2,2) & (2,3) & \cdots & (2,n) \\
(3,1) & (3,2) & (3,3) & \cdots & (2,n) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
(n,1) & (n,2) & (n,3) & \cdots & (n,n)
\end{matrix}
$$

Way 1: Combinatorial argument

$$\binom{n}{2}$$

Way 2: Geometric argument

$$2 \times [Triangle] + [Diagonal] = n^2$$

$$[Triangle] = (n^2 - n)/2$$

$$= \frac{\binom{n}{2}}{\mathrm{n}^2}$$

Problem 3: Sampling

$$\begin{matrix} (1,1) & (1,2) & (1,3) & \cdots & (1,n) \\ (2,1) & (2,2) & (2,3) & \cdots & (2,n) \\ (3,1) & (3,2) & (3,3) & \cdots & (2,n) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (n,1) & (n,2) & (n,3) & \cdots & (n,n) \end{matrix}$$

$$\{(1,2),(2,3),(3,4),\ldots(n-1,n)\}$$

$$= \frac{n-1}{\mathrm{n}^2}$$

Problem 3: Sampling

$$\begin{matrix} (1,1) & (1,2) & (1,3) & \cdots & (1,n) \\ (2,1) & (2,2) & (2,3) & \cdots & (2,n) \\ (3,1) & (3,2) & (3,3) & \cdots & (2,n) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (n,1) & (n,2) & (n,3) & \cdots & (n,n) \end{matrix}$$

$(1,2)$

$$\frac{1}{n} \cdot \frac{1}{(n-1)} = \frac{1}{n(n-1)}$$

$$\frac{1}{n(n-1)}$$

$$\text{(b)} = \frac{1}{2} \qquad (c) = \frac{(n-1)}{n(n-1)}$$

Problem 4: Intransitive Dice

What is the sample space and probability function?

$(2,1)$	$(4,1)$	$(9,1)$
$(2,6)$	$(4,6)$	$(9,6)$
$(2,8)$	$(4,8)$	$(9,8)$

$\frac{1}{9}$

$$\mathbb{P}[you\ win] = \frac{5}{9}$$

Problem 4: Intrasitive Dice

(b) $\mathbb{P}[you\ win] = \frac{5}{9}$

(c) $\mathbb{P}[you\ win] = \frac{5}{9}$

(d) Going second guarantees higher probability of winning