Counting

CS 70 Discussion 5B

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Counting Intro

	with replacement	without replacement
order matters	n^k	$\frac{n!}{(n-k)!}$
order doesn't matter	$\binom{n+k-1}{k} = \binom{n+k-1}{n-1}$	$\frac{n!}{k!(n-k)!} = \binom{n}{k}$

Counting Intro

Order matters, with replacement

How many 2 letter sequence can be picked from the set $\{1, 2, 3, 4\}$, with replacement

$\{1, 1\}$	$\{1, 2\}$	$\{1, 3\}$	$\{1, 4\}$
$\{2, 1\}$	$\{2, 2\}$	$\{2, 3\}$	$\{2, 4\}$
$\{3, 1\}$	$\{3, 2\}$	$\{3, 3\}$	$\{3, 4\}$
$\{4, 1\}$	$\{4, 2\}$	$\{4, 3\}$	$\{4, 4\}$

$$\underline{\quad}\ \underline{\quad}$$

$$4 \cdot 4 = 16$$

$$\underline{\quad}\ \underline{\quad}\ \dots\ \underline{\quad}\ \underline{\quad}$$

$$n \cdot n \cdot \dots\ n \cdot n = n^k$$

Counting Intro

Order matters, without replacement

How many 2 letter sequence can be picked from the set $\{1, 2, 3, 4\}$, without replacement

$\{1,1\}$	$\{1,2\}$	$\{1,3\}$	$\{1,4\}$
$\{2,1\}$	$\{2,2\}$	$\{2,3\}$	$\{2,4\}$
$\{3,1\}$	$\{3,2\}$	$\{3,3\}$	$\{3,4\}$
$\{4,1\}$	$\{4,2\}$	$\{4,3\}$	$\{4,4\}$

$$\underline{\quad 4 \quad} \cdot \underline{\quad 3 \quad} = 12$$

$$\underline{\quad n \quad} \cdot \underline{(n-1)} \cdot \underline{(n-2)} \cdots \underline{(n-k+1)}$$

$$= \frac{n!}{(n-k)!}$$

Problem 1: Counting Intro I

(a)

___ ___ ___ ___

52 51 50 49

$$= \frac{n!}{(n-k)!} \cdot \frac{1}{k!}$$ Overcounting

Permutation

4! ways

$\{A, 2, 3, 4\}$	$\{2, A, 3, 4\}$	$\{3, A, 2, 4\}$	$\{4, A, 2, 3\}$
$\{A, 2, 4, 3\}$	$\{2, A, 4, 3\}$	$\{3, A, 4, 2\}$	$\{4, A, 3, 2\}$
$\{A, 3, 2, 4\}$	$\{2, 3, A, 4\}$	$\{3, 2, A, 4\}$	$\{4, 2, A, 3\}$
$\{A, 3, 4, 2\}$	$\{2, 3, A, 2\}$	$\{3, 2, A, 2\}$	$\{4, 2, 3, A\}$
$\{A, 4, 2, 3\}$	$\{2, 4, A, 3\}$	$\{3, 4, A, 2\}$	$\{4, 3, A, 2\}$
$\{A, 4, 3, 2\}$	$\{2, 4, A, 2\}$	$\{3, 4, 2, A\}$	$\{4, 3, 2, A\}$

$[A, 2, 3, 4]$

Permutation space $= \frac{52!}{48!}$

Combination space $= \frac{52!}{48!} \cdot \frac{1}{4!}$

Problem 1: Counting Intro I

(b)

$$\underset{+1}{\_\_}\quad \underset{10}{\_\_}\ \underset{10}{\_\_}\ \underset{10}{\_\_}\quad \underset{10}{\_\_}\ \underset{10}{\_\_}\ \underset{10}{\_\_}\quad \underset{10}{\_\_}\ \underset{10}{\_\_}\ \underset{10}{\_\_}\ \underset{10}{\_\_} = 10^{10}$$

(c) Selecting from the set $\{C, O, V, E, R\}$

$$\underset{5}{\_\_}\ \underset{4}{\_\_}\ \underset{3}{\_\_}\ \underset{2}{\_\_}\ \underset{1}{\_\_} = 5!$$

Easier example, anagram of AAB, i.e. selecting from the set $\{A, A, B\} = \{A_1, A_2, B\}$

A_1A_2B	A_2A_1B	AAB
A_1BA_2	A_2BA_1	ABA
BA_1A_2	BA_2A_1	BAA

$$= \frac{3!}{2!}$$

Problem 1: Counting Intro I

(c) $\{B, E, R, K, E, L, E, Y\}$

$$= \frac{8!}{3!}$$

What about $\{B, B, E, R, K, E, L, E, Y\}$

$$= \frac{8!}{2! \cdot 3!}$$

Permuting B → $2!$; $3!$ ← Permuting E

Another perspective:

1	2	3	4	5	6	7	8
E		E	E	B	B		

Step 1: Pick 3 index for the 3 Es $\frac{8!}{3!\,5!}$

Step 2: Pick 2 index for the 2 Bs $\frac{5!}{2!\,3!}$

Step 2: Permute the remainings $3!$

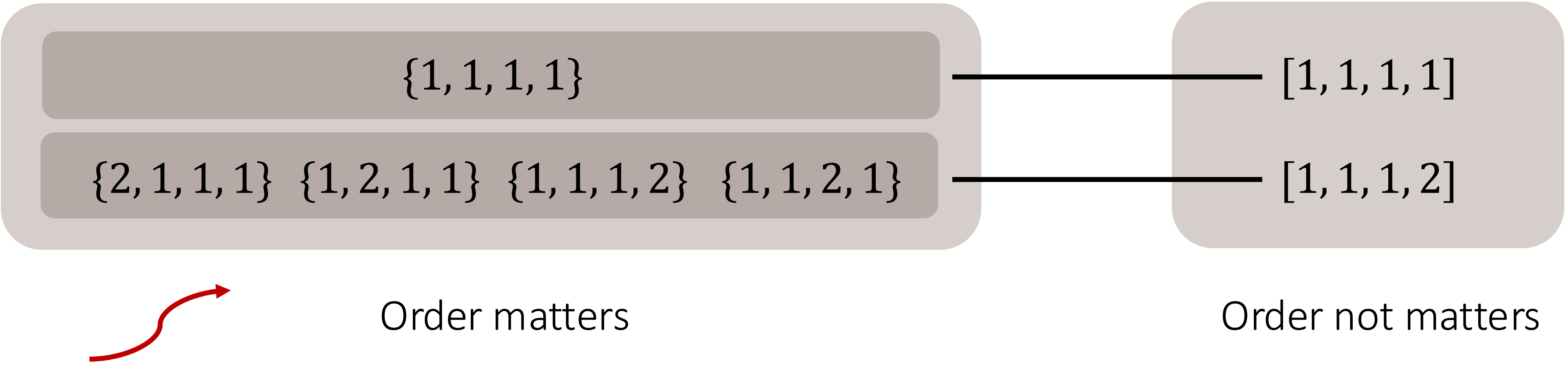
$$= \frac{8!}{3!\,5!} \cdot \frac{5!}{2!\,3!} \cdot 3! = \frac{8!}{2!\,3!}$$

Problem 1: Counting Intro I

(d) Choose 10 numbers from the set $\{1, 2, 3, 4\}$, with replacement, order doesn't matter

___ ___ ___ ___ ___ ___ ___ ___ ___ ___

$4 \quad 4 \quad 4 \quad 4 \quad 4 \quad 4 \quad 4 \quad 4 \quad 4 \quad 4$



The size is different!

Problem 1: Counting Intro I

(d) Choose 10 numbers from the set $\{1, 2, 3, 4\}$, with replacement, order doesn't matter

Order doesn't matter $\implies$ Can just store "counts"

$$[1,1,1,1,1,1,1,1,1,1] \quad \Rightarrow \quad \{1{:}\,10, 2{:}\,0, 3{:}\,0, 4{:}\,0\}$$

$$[1,2,2,2,2,3,3,3,4,4] \quad \Rightarrow \quad \{1{:}\,1, 2{:}\,4, 3{:}\,3, 4{:}\,2\}$$

$$[1,2,2,2,2,4,4,4,4,4] \quad \Rightarrow \quad \{1{:}\,1, 2{:}\,4, 3{:}\,0, 4{:}\,5\}$$

Let x_1, x_2, x_3, x_4 be the number of $1, 2\ , 3, 4$ respectively

Count how many nonnegative solutions does

$$x_1 + x_2 + x_3 + x_4 = 10$$

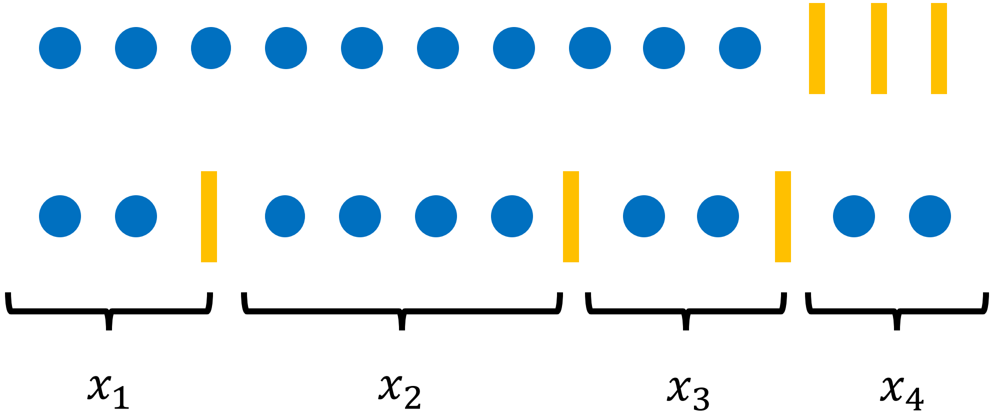
Has?

Problem 1: Counting Intro I

(d) Count the number of nonnegative solutions:

$$x_1 + x_2 + x_3 + x_4 = 10$$

Consider 10 balls and 3 sticks, each configuration bijectively identifies to a solution



$$\Longleftrightarrow \quad x_1 = 10, x_2 = 0, x_3 = 0, x_4 = 0$$

$$\Longleftrightarrow \quad x_1 = 2, x_2 = 4, x_3 = 2, x_4 = 2$$

$$= \frac{13!}{10!\,3!}$$

Problem 3: Farmer's Market

(a) Choose k elements from the set $\{A, P\}$

Same as counting the number of nonnegative solutions to

$$x_A + x_P = k$$

$$= \frac{(k+1)!}{k!\,1!} = k + 1$$

(b) Choose k elements from the set $\{A, P, O, E\}$

Same as counting the number of nonnegative solutions to

$$x_A + x_P + x_O + x_E = k$$

$$= \frac{(k+3)!}{k!\,3!}$$

Problem 3: Farmer's Market

(c) Choose k elements from the set $\{1, 2, 3, \dots, n\}$

Same as counting the number of nonnegative solutions to

$$x_1 + x_2 + \cdots + x_n = k$$

$$= \frac{(k+n-1)!}{k!\,(n-1)!}$$

Issue: We need at least two different kinds of fruits

$$= \frac{(k+n-1)!}{k!\,(n-1)!} - \boxed{n}$$

We don't want cases like

$[1, 1, \dots, 1]$
$[2, 2, \dots, 2]$
…
$[n, n, \dots, n]$

Problem 4: The Count

(a) Sample with replacement, order not matter from the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

For each sample there is only one way of arranging them in non-increasing manner

$$[9, 3, 2, 1, 0, 0, 0]$$

$$= \frac{(10 + 7 - 1)!}{7!\,(10 - 1)!} = \frac{16!}{7!\,9!}$$

(b) Sample without replacement, order not matter from the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

$$= \frac{10!}{7!\,(10 - 7)!} = \frac{10!}{7!\,3!}$$

Problem 4: The Count

(c) Casework on where the sequence of 0s start

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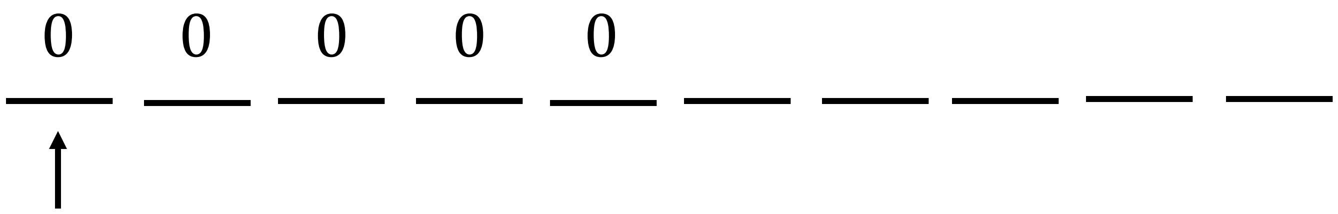
(c) Casework on where the sequence of 0s start

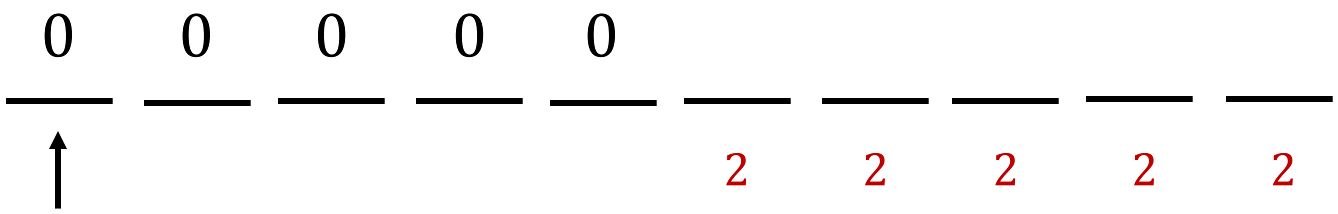
Suppose start at index 0



Problem 4: The Count

(c) Casework on where the sequence of 0s start

Suppose start at index 0



Problem 4: The Count

(c) Casework on where the sequence of 0s start

Suppose start at index 0

0	0	0	0	0						
					2	2	2	2	2	$= 2^5 = 32$

Problem 4: The Count

(c) Casework on where the sequence of 0s start

Suppose start at index 0

0	0	0	0	0						
↑					2	2	2	2	2	$= 2^5 = 32$

Suppose start at index 1

	0	0	0	0	0				
	↑								

Problem 4: The Count

(c) Casework on where the sequence of 0s start

Suppose start at index 0

0	0	0	0	0						
↑					2	2	2	2	2	$= 2^5 = 32$

Suppose start at index 1

	0	0	0	0	0				
2	↑					2	2	2	2

Problem 4: The Count

(c) Casework on where the sequence of 0s start

Suppose start at index 0

0	0	0	0	0						
↑					2	2	2	2	2	$= 2^5 = 32$

Suppose start at index 1

0	0	0	0	0	0				
2	↑					2	2	2	2

Problem 4: The Count

(c) Casework on where the sequence of 0s start

Suppose start at index 0

0 0 0 0 0 _ _ _ _ _

↑ 2 2 2 2 2 $= 2^5 = 32$

Suppose start at index 1

0 0 0 0 0 0 _ _ _ _

2 ↑ 2 2 2 2

If the first digit is 0, then we've overcounted!

Problem 4: The Count

(c) Casework on where the sequence of 0s start

Suppose start at index 0

$$\underline{0}\ \underline{0}\ \underline{0}\ \underline{0}\ \underline{0}\ \underline{\ 2\ }\ \underline{\ 2\ }\ \underline{\ 2\ }\ \underline{\ 2\ }\ \underline{\ 2\ } = 2^5 = 32$$

Suppose start at index 1

$$\underline{1}\ \underline{0}\ \underline{0}\ \underline{0}\ \underline{0}\ \underline{0}\ \underline{\ 2\ }\ \underline{\ 2\ }\ \underline{\ 2\ }\ \underline{\ 2\ }$$

If the first digit is 0, then we've overcounted!

Problem 4: The Count

(c) Casework on where the sequence of 0s start

Suppose start at index 0

$0\ 0\ 0\ 0\ 0\ \_\ \_\ \_\ \_\ \_$

$2\ 2\ 2\ 2\ 2 = 2^5 = 32$

Suppose start at index 1

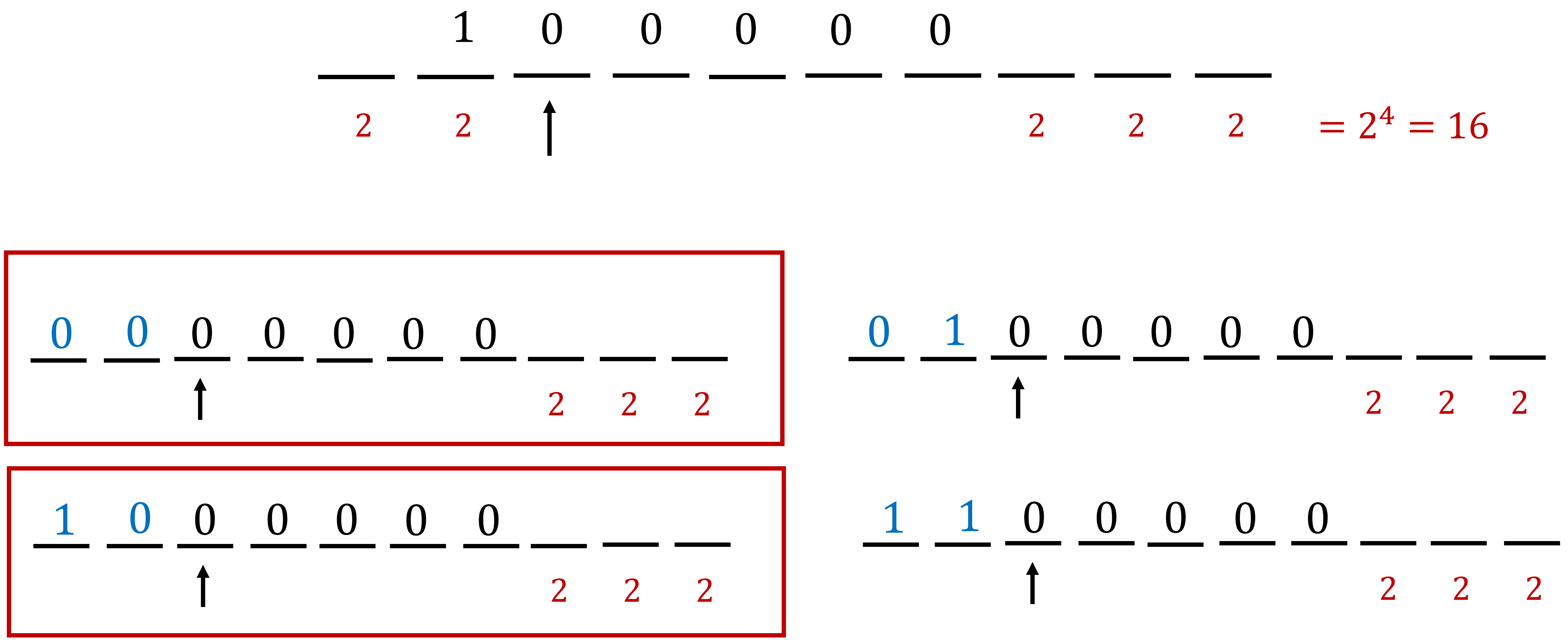
$1\ 0\ 0\ 0\ 0\ 0\ \_\ \_\ \_\ \_$

$2\ 2\ 2\ 2 = 2^4 = 16$

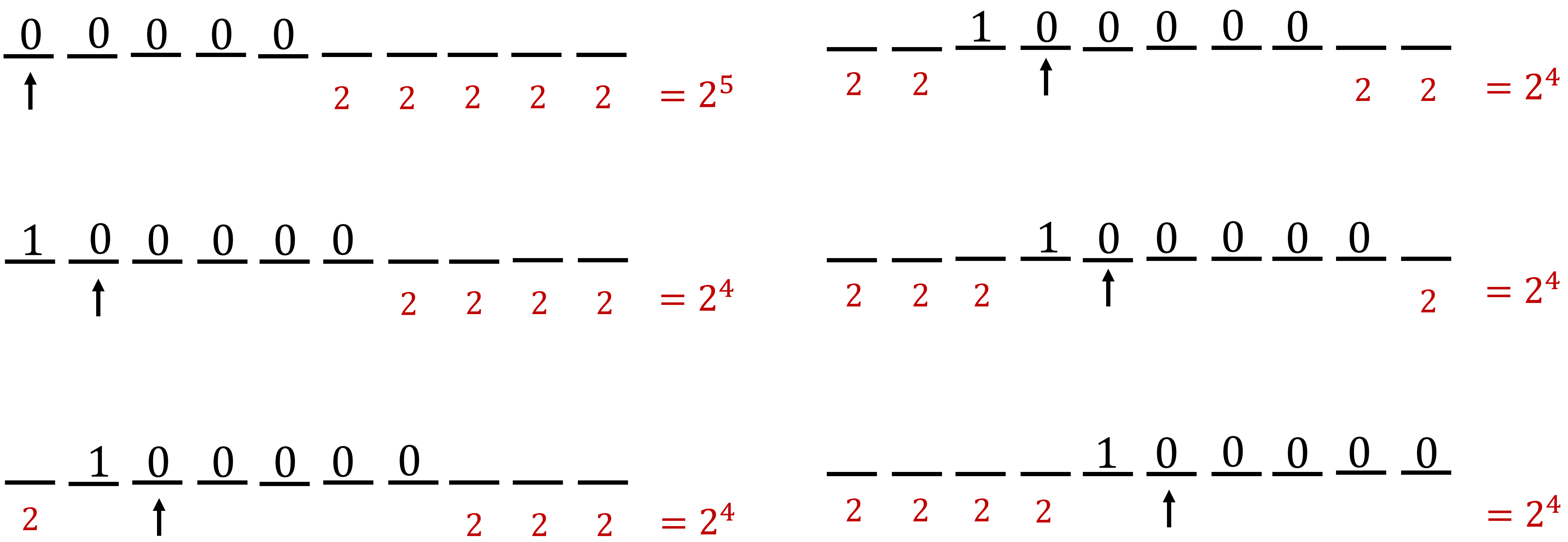
If the first digit is 0, then we've overcounted!

Problem 4: The Count

Suppose start at index **2**



Problem 4: The Count



$$= 2^5 + 5 \cdot 2^4$$