Graph I

CS 70 Discussion 2A

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Note: These slides are unofficial course materials. Please use the notes as the only single source of truth.

Problem 1: Degree Sequence

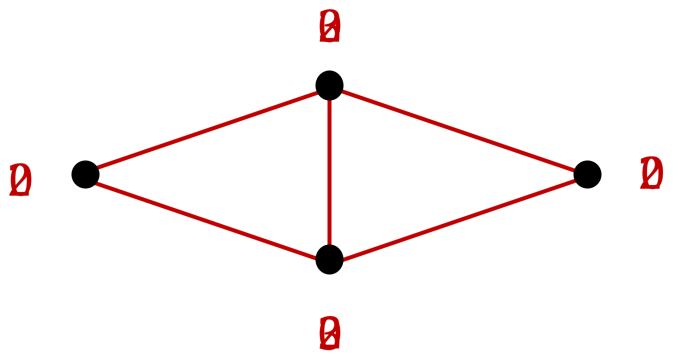
a) $(3, 3, 2, 2)$

First step: Check using handshaking lemma

Note (Handshaking Lemma):

$$G = (V, E) \text{ is a graph} \quad \Longrightarrow \quad \sum_{v \in V} \deg(v) = 2|E|$$

Sum of degree is even, so it's "possible" that there exists such graphs



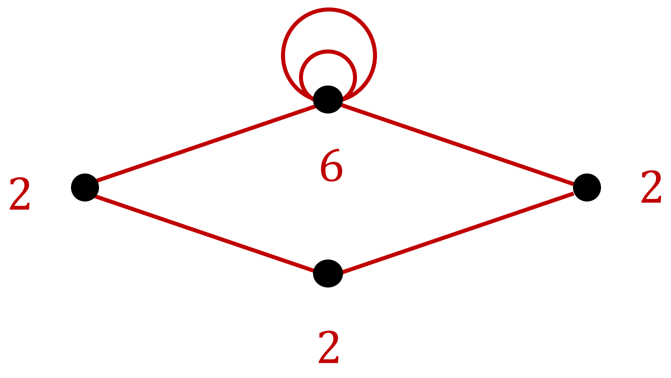
Problem 1: Degree Sequence

b) $(3, 3, 2, 2, 2, 2, 1, 1)$

Sum of degree is odd, so no such graphs exist.

c) $(6, 2, 2, 2)$

Sum of degree is even, so it's possible such graphs exist.

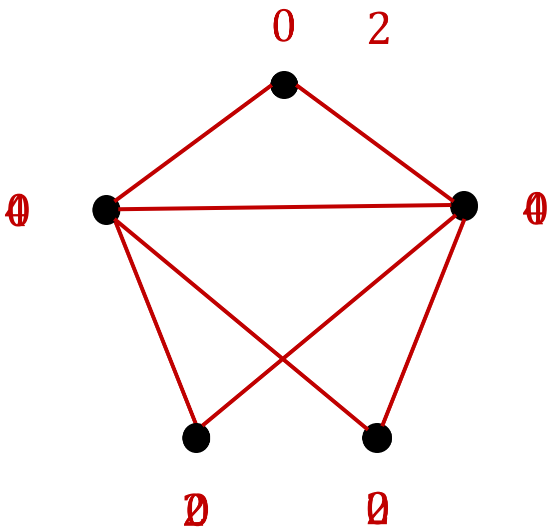


Total number of vertex is 4, so cannot have degree 6 vertex!

Problem 1: Degree Sequence

d) $(4, 4, 3, 2, 1)$

Sum of degree is even, so it's "possible" such graphs exist.



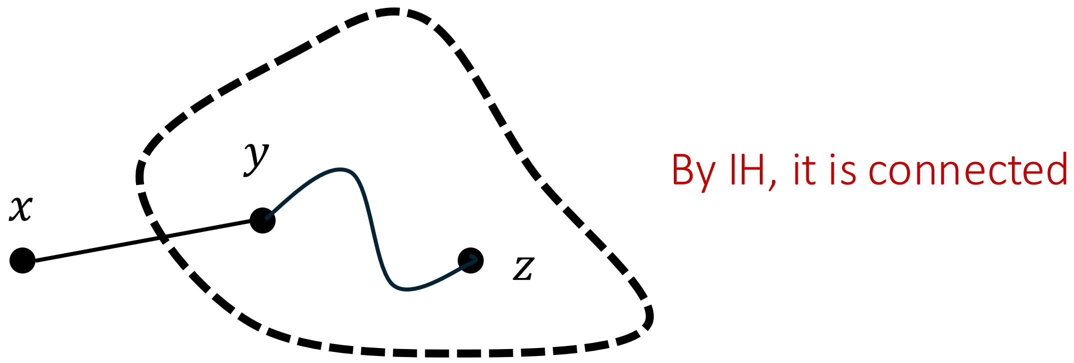
Impossible! The other remaining vertex must have minimum degree of 2!

Problem 2: Build-Up Error

False claim: If every vertex in an undirected graph with $|V| \geq 2$ has degree at least 1, then it is connected

Induction hypothesis: Suppose the statement is true for some $n \geq 2$.

Induction step: We prove the claim for $n + 1$. Consider any undirected graph on n vertices, which every vertex has degree at least 1.

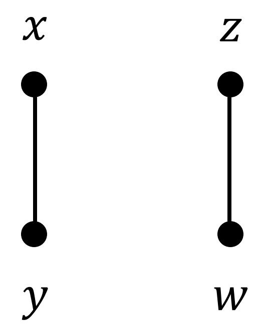


n vertex graph

Problem 2: Build-Up Error

Issue: Not every graph with $n+1$ vertices can be constructed in such way

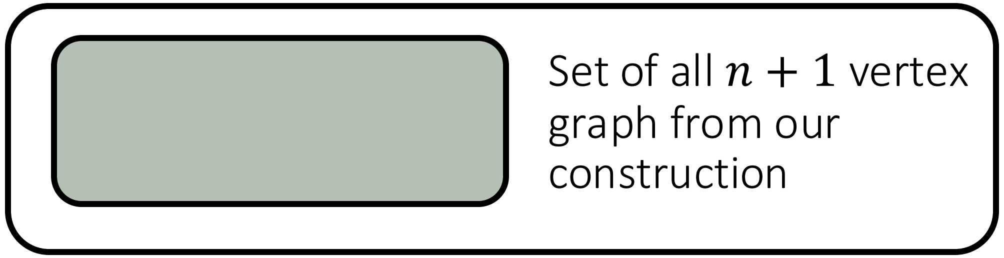
Example:



It's impossible to construct this graph from a 3 vertex graph with min degree 1!

Problem 2: Build-Up Error

Dangerous: start from n-vertex graph and build $n + 1$-vertex graph



Set of all $n + 1$ vertex graph

Need to argue that these two sets are the same!

Safer: Start from $n + 1$-vertex graph and remove some vertex to get to n-vertex graph

Then apply induction hypothesis

Problem 3: Eulerian Tour and Walk

a) Is there an Eulerian tour in the given graph?

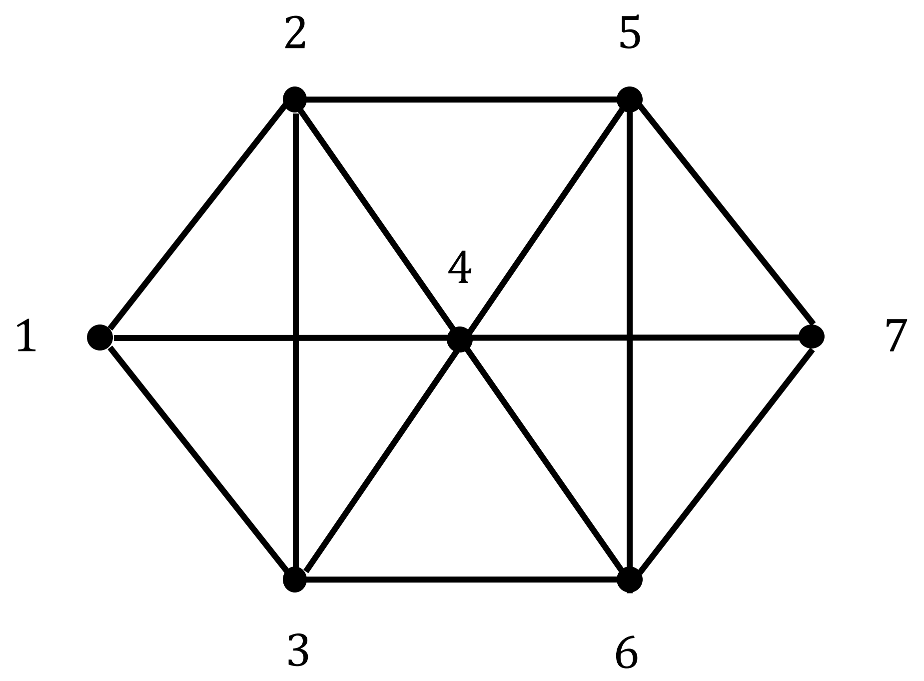
Note (Euler's theorem):

A graph $G = (V, E)$ has an Eulerian tour $\iff$ G is connected and even degree

So no, G is not even degree

Problem 3: Eulerian Tour and Walk

b) Is there an Eulerian walk in the graph below?



Yes: $1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 1 \rightarrow 3 \rightarrow 6 \rightarrow 4 \rightarrow 5 \rightarrow 2 \rightarrow 4 \rightarrow 7 \rightarrow 5 \rightarrow 6 \rightarrow 7$

Problem 3: Eulerian Tour and Walk

c) What is the condition that there is an Eulerian walk in an undirected graph?

A graph G has an Eulerian walk $\iff$ G is connected and has ≤ 2 odd degree vertex

$\Longrightarrow$):

Suppose G has an Eulerian walk

$\Longrightarrow$ Connected, and …

$\Longrightarrow$ Only the last two vertex is odd degree

Problem 3: Eulerian Tour and Walk

$\Longleftarrow$):

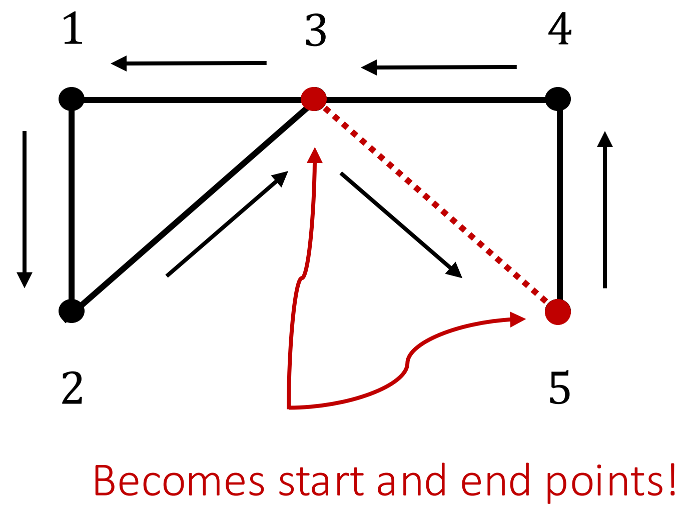
Suppose G is connected and has ≤ 2 odd degree vertex

Connect the two odd degree vertex

$\Longrightarrow$ G is now connected and even degree

$\Longrightarrow$ G has an Eulerian tour

Removing the added edge gives an Eulerian walk!



Problem 4: Coloring Trees

a) Prove that all trees with at least 2 vertices have at least two leaves. Recall that a leaf is defined as a node in a tree with degree exactly 1.

Proof 1 (Handshaking Lemma).

$$2(n-1) = \sum_{v \in V} \deg(v) = \sum_{v \in L} \underbrace{\boxed{\deg(v)}}_{=1} + \sum_{v \in V \setminus L} \underbrace{\boxed{\deg(v)}}_{\geq 2}$$

$$\geq \sum_{v \in L} 1 + \sum_{v \in V \setminus L} 2 = |\mathrm{L}| + 2(\mathrm{n} - |\mathrm{L}|)$$

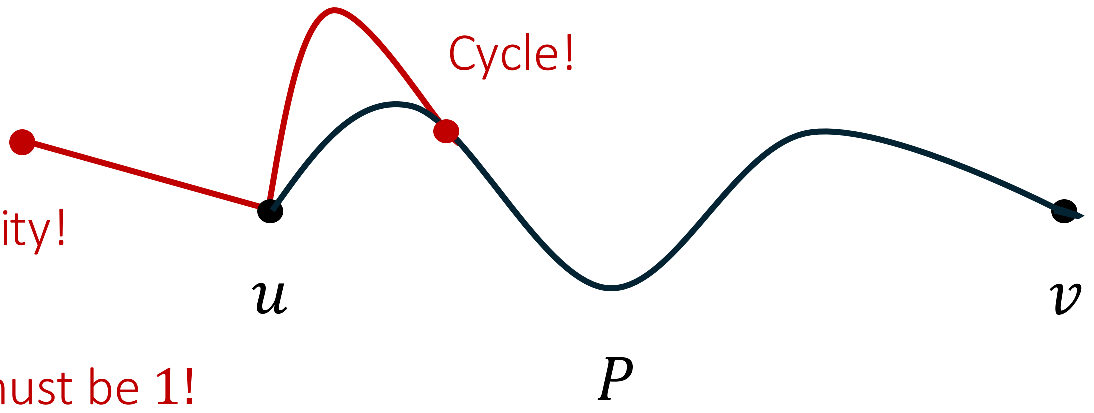
$$= 2n - |L|$$

Problem 4: Coloring Trees

a) Prove that all trees with at least 2 vertices have at least two leaves. Recall that a leaf is defined as a node in a tree with degree exactly 1.

Proof 2 (Maximal path).

Consider the maximal path P in the tree, with endpoints u, v.



Violates maximality!

The degree of u, v must be 1!

Problem 4: Coloring Trees

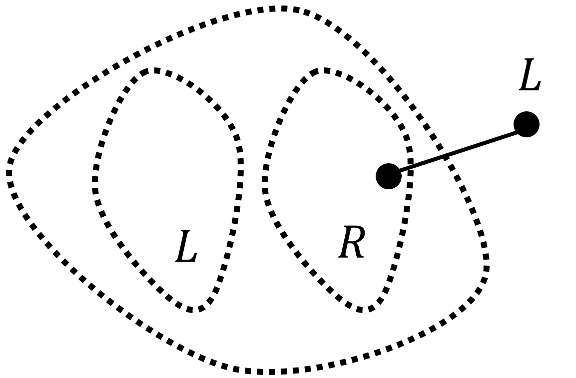
b) Prove that all trees with at least 2 vertices are bipartite.

Proof 1 (Induction).

Induction hypothesis: Suppose the statement is true for trees with n vertex

Induction step: Consider any tree with $n + 1$ vertex

- Remove a leaf
- Apply IH on the n vertex tree
- Reassign the last vertex based on where it's connected



$n + 1$ vertex tree

Problem 4: Coloring Trees

b) Prove that all trees with at least 2 vertices are bipartite.

Proof 2 (BFS).

Use BFS to color the vertices!