# Stable Matching

CS 70 Discussion 1B

Raymond Tsao

2025-01-24

Note: These slides are unofficial course materials. Please use the notes as the only single source of truth.

# Problem 1

| $J$ | $C$ | | |
|---|---|---|---|
| $1$ | $A$ | $B$ | $C$ |
| $2$ | $B$ | $A$ | $C$ |
| $3$ | $A$ | $B$ | $C$ |

| $C$ | $J$ | | |
|---|---|---|---|
| $A$ | $2$ | $1$ | $3$ |
| $B$ | $1$ | $3$ | $2$ |
| $C$ | $1$ | $2$ | $3$ |

**Note (Gale Shapely Algorithm):**

On each iteration (day)

- Step 1: Each job proposes to its favorite
- Step 2: Each candidate reject all but their favorite offers.
- Step 3: Each rejected job crosses off the candidate who rejected its offer from its list

# Problem 1

| $J$ | $C$ | | |
|---|---|---|---|
| $1$ | $A$ | $B$ | $C$ |
| $2$ | $B$ | $A$ | $C$ |
| $3$ | $A$ | $B$ | $C$ |

| $C$ | $J$ | | |
|---|---|---|---|
| $A$ | $2$ | $1$ | $3$ |
| $B$ | $1$ | $3$ | $2$ |
| $C$ | $1$ | $2$ | $3$ |

| $C$ | | | | | |
|---|---|---|---|---|---|
| $A$ | | | | | |
| $B$ | | | | | |
| $C$ | | | | | |

# Problem 1

| $J$ | $C$ | | |
|---|---|---|---|
| $1$ | $A$ | $B$ | $C$ |
| $2$ | $B$ | $A$ | $C$ |
| $3$ | $A$ | $B$ | $C$ |

| $C$ | $J$ | | |
|---|---|---|---|
| $A$ | $2$ | $1$ | $3$ |
| $B$ | $1$ | $3$ | $2$ |
| $C$ | $1$ | $2$ | $3$ |

| $C$ | $Day\ 1$ | | | | |
|---|---|---|---|---|---|
| $A$ | | | | | |
| $B$ | | | | | |
| $C$ | | | | | |

# Problem 1

| $J$ | $C$ | | |
|---|---|---|---|
| $1$ | $A$ | $B$ | $C$ |
| $2$ | $B$ | $A$ | $C$ |
| $3$ | $A$ | $B$ | $C$ |

| $C$ | $J$ | | |
|---|---|---|---|
| $A$ | $2$ | $1$ | $3$ |
| $B$ | $1$ | $3$ | $2$ |
| $C$ | $1$ | $2$ | $3$ |

| $C$ | $Day\ 1$ | | | | |
|---|---|---|---|---|---|
| $A$ | | | | | |
| $B$ | | | | | |
| $C$ | | | | | |

# Problem 1

| $J$ | $C$ | | |
|---|---|---|---|
| $1$ | $A$ | $B$ | $C$ |
| $2$ | $B$ | $A$ | $C$ |
| $3$ | $A$ | $B$ | $C$ |

| $C$ | $J$ | | |
|---|---|---|---|
| $A$ | $2$ | $1$ | $3$ |
| $B$ | $1$ | $3$ | $2$ |
| $C$ | $1$ | $2$ | $3$ |

| $C$ | $Day\ 1$ | | | | |
|---|---|---|---|---|---|
| $A$ | $1, 3$ | | | | |
| $B$ | $2$ | | | | |
| $C$ | | | | | |

Step 1: Job propose

# Problem 1

| $J$ | $C$ | | |
|---|---|---|---|
| $1$ | $A$ | $B$ | $C$ |
| $2$ | $B$ | $A$ | $C$ |
| $3$ | $A$ | $B$ | $C$ |

| $C$ | $J$ | | |
|---|---|---|---|
| $A$ | $2$ | $1$ | $3$ |
| $B$ | $1$ | $3$ | $2$ |
| $C$ | $1$ | $2$ | $3$ |

| $C$ | $Day\ 1$ | | | | |
|---|---|---|---|---|---|
| $A$ | 1, ~~3~~ | | | | |
| $B$ | 2 | | | | |
| $C$ | | | | | |

Step 2: Candidate reject

# Problem 1

| $J$ | $C$ | | |
|---|---|---|---|
| $1$ | $A$ | $B$ | $C$ |
| $2$ | $B$ | $A$ | $C$ |
| $3$ | ~~$A$~~ | $B$ | $C$ |

| $C$ | $J$ | | |
|---|---|---|---|
| $A$ | $2$ | $1$ | $3$ |
| $B$ | $1$ | $3$ | $2$ |
| $C$ | $1$ | $2$ | $3$ |

| $C$ | $Day\ 1$ | | | | |
|---|---|---|---|---|---|
| $A$ | 1,~~3~~ | | | | |
| $B$ | 2 | | | | |
| $C$ | | | | | |

Step 3: Job update preference

# Problem 1

| $J$ | $C$ | | |
|---|---|---|---|
| $1$ | $A$ | $B$ | $C$ |
| $2$ | $B$ | $A$ | $C$ |
| $3$ | ~~$A$~~ | $B$ | $C$ |

| $C$ | $J$ | | |
|---|---|---|---|
| $A$ | $2$ | $1$ | $3$ |
| $B$ | $1$ | $3$ | $2$ |
| $C$ | $1$ | $2$ | $3$ |

| $C$ | $Day\ 1$ | $Day\ 2$ | | | |
|---|---|---|---|---|---|
| $A$ | 1,~~3~~ | | | | |
| $B$ | 2 | | | | |
| $C$ | | | | | |

# Problem 1

| $J$ | $C$ | | |
|---|---|---|---|
| $1$ | $A$ | $B$ | $C$ |
| $2$ | $B$ | $A$ | $C$ |
| $3$ | ~~$A$~~ | $B$ | $C$ |

| $C$ | $J$ | | |
|---|---|---|---|
| $A$ | $2$ | $1$ | $3$ |
| $B$ | $1$ | $3$ | $2$ |
| $C$ | $1$ | $2$ | $3$ |

| $C$ | $Day\ 1$ | $Day\ 2$ | | | |
|---|---|---|---|---|---|
| $A$ | $1,$~~$3$~~ | $1$ | | | |
| $B$ | $2$ | $2, 3$ | | | |
| $C$ | | | | | |

Step 1: Job propose

# Problem 1

| $J$ | $C$ | | |
|---|---|---|---|
| $1$ | $A$ | $B$ | $C$ |
| $2$ | $B$ | $A$ | $C$ |
| $3$ | ~~$A$~~ | $B$ | $C$ |

| $C$ | $J$ | | |
|---|---|---|---|
| $A$ | $2$ | $1$ | $3$ |
| $B$ | $1$ | $3$ | $2$ |
| $C$ | $1$ | $2$ | $3$ |

| $C$ | $Day\ 1$ | $Day\ 2$ | | | |
|---|---|---|---|---|---|
| $A$ | $1,$~~$3$~~ | $1$ | | | |
| $B$ | $2$ | ~~$2$~~, $3$ | | | |
| $C$ | | | | | |

Step 2: Candidate reject

# Problem 1

| $J$ | $C$ | | |
|---|---|---|---|
| $1$ | $A$ | $B$ | $C$ |
| $2$ | ~~$B$~~ | $A$ | $C$ |
| $3$ | ~~$A$~~ | $B$ | $C$ |

| $C$ | $J$ | | |
|---|---|---|---|
| $A$ | $2$ | $1$ | $3$ |
| $B$ | $1$ | $3$ | $2$ |
| $C$ | $1$ | $2$ | $3$ |

| $C$ | $Day\ 1$ | $Day\ 2$ | | | |
|---|---|---|---|---|---|
| $A$ | 1,~~3~~ | 1 | | | |
| $B$ | 2 | ~~2~~, 3 | | | |
| $C$ | | | | | |

Step 3: Job update preference

# Problem 1

| $J$ | $C$ | | |
|---|---|---|---|
| $1$ | $A$ | $B$ | $C$ |
| $2$ | ~~$B$~~ | $A$ | $C$ |
| $3$ | ~~$A$~~ | $B$ | $C$ |

| $C$ | $J$ | | |
|---|---|---|---|
| $A$ | $2$ | $1$ | $3$ |
| $B$ | $1$ | $3$ | $2$ |
| $C$ | $1$ | $2$ | $3$ |

| $C$ | $Day\ 1$ | $Day\ 2$ | $Day\ 3$ | | |
|---|---|---|---|---|---|
| $A$ | 1,~~3~~ | 1 | | | |
| $B$ | 2 | ~~2~~, 3 | | | |
| $C$ | | | | | |

# Problem 1

| $J$ | $C$ | | |
|---|---|---|---|
| $1$ | $A$ | $B$ | $C$ |
| $2$ | ~~$B$~~ | $A$ | $C$ |
| $3$ | ~~$A$~~ | $B$ | $C$ |

| $C$ | $J$ | | |
|---|---|---|---|
| $A$ | $2$ | $1$ | $3$ |
| $B$ | $1$ | $3$ | $2$ |
| $C$ | $1$ | $2$ | $3$ |

| $C$ | $Day\ 1$ | $Day\ 2$ | $Day\ 3$ | | |
|---|---|---|---|---|---|
| $A$ | $1,$~~$3$~~ | $1$ | $1, 2$ | | |
| $B$ | $2$ | ~~$2$~~$, 3$ | $3$ | | |
| $C$ | | | | | |

Step 1: Job propose

# Problem 1

| $J$ | $C$ | | |
|---|---|---|---|
| $1$ | $A$ | $B$ | $C$ |
| $2$ | ~~$B$~~ | $A$ | $C$ |
| $3$ | ~~$A$~~ | $B$ | $C$ |

| $C$ | $J$ | | |
|---|---|---|---|
| $A$ | $2$ | $1$ | $3$ |
| $B$ | $1$ | $3$ | $2$ |
| $C$ | $1$ | $2$ | $3$ |

| $C$ | $Day\ 1$ | $Day\ 2$ | $Day\ 3$ | | |
|---|---|---|---|---|---|
| $A$ | 1, ~~3~~ | 1 | ~~1~~, 2 | | |
| $B$ | 2 | ~~2~~, 3 | 3 | | |
| $C$ | | | | | |

Step 2: Candidate reject

# Problem 1

| $J$ | $C$ | | |
|---|---|---|---|
| $1$ | ~~$A$~~ | $B$ | $C$ |
| $2$ | ~~$B$~~ | $A$ | $C$ |
| $3$ | ~~$A$~~ | $B$ | $C$ |

| $C$ | $J$ | | |
|---|---|---|---|
| $A$ | $2$ | $1$ | $3$ |
| $B$ | $1$ | $3$ | $2$ |
| $C$ | $1$ | $2$ | $3$ |

| $C$ | $Day\ 1$ | $Day\ 2$ | $Day\ 3$ | | |
|---|---|---|---|---|---|
| $A$ | 1, ~~3~~ | 1 | ~~1~~, 2 | | |
| $B$ | 2 | ~~2~~, 3 | 3 | | |
| $C$ | | | | | |

Step 3: Job update preference

# Problem 1

| $J$ | $C$ | | |
|---|---|---|---|
| $1$ | ~~$A$~~ | $B$ | $C$ |
| $2$ | ~~$B$~~ | $A$ | $C$ |
| $3$ | ~~$A$~~ | $B$ | $C$ |

| $C$ | $J$ | | |
|---|---|---|---|
| $A$ | $2$ | $1$ | $3$ |
| $B$ | $1$ | $3$ | $2$ |
| $C$ | $1$ | $2$ | $3$ |

| $C$ | $Day\ 1$ | $Day\ 2$ | $Day\ 3$ | $Day\ 4$ | |
|---|---|---|---|---|---|
| $A$ | 1, ~~3~~ | 1 | ~~1~~, 2 | | |
| $B$ | 2 | ~~2~~, 3 | 3 | | |
| $C$ | | | | | |

# Problem 1

| $J$ | $C$ | | |
|---|---|---|---|
| $1$ | ~~$A$~~ | $B$ | $C$ |
| $2$ | ~~$B$~~ | $A$ | $C$ |
| $3$ | ~~$A$~~ | $B$ | $C$ |

| $C$ | $J$ | | |
|---|---|---|---|
| $A$ | $2$ | $1$ | $3$ |
| $B$ | $1$ | $3$ | $2$ |
| $C$ | $1$ | $2$ | $3$ |

| $C$ | $Day\ 1$ | $Day\ 2$ | $Day\ 3$ | $Day\ 4$ | |
|---|---|---|---|---|---|
| $A$ | $1,$~~$3$~~ | $1$ | ~~$1$~~$, 2$ | $2$ | |
| $B$ | $2$ | ~~$2$~~$, 3$ | $3$ | $1, 3$ | |
| $C$ | | | | | |

Step 1: Job propose

# Problem 1

| $J$ | $C$ | | |
|---|---|---|---|
| $1$ | ~~$A$~~ | $B$ | $C$ |
| $2$ | ~~$B$~~ | $A$ | $C$ |
| $3$ | ~~$A$~~ | $B$ | $C$ |

| $C$ | $J$ | | |
|---|---|---|---|
| $A$ | $2$ | $1$ | $3$ |
| $B$ | $1$ | $3$ | $2$ |
| $C$ | $1$ | $2$ | $3$ |

| $C$ | $Day\ 1$ | $Day\ 2$ | $Day\ 3$ | $Day\ 4$ | |
|---|---|---|---|---|---|
| $A$ | $1,$~~$3$~~ | $1$ | ~~$1$~~$, 2$ | $2$ | |
| $B$ | $2$ | ~~$2$~~$, 3$ | $3$ | $1,$~~$3$~~ | |
| $C$ | | | | | |

Step 2: Candidate reject

# Problem 1

| $J$ | $C$ | | |
|---|---|---|---|
| $1$ | ~~$A$~~ | $B$ | $C$ |
| $2$ | ~~$B$~~ | $A$ | $C$ |
| $3$ | ~~$A$~~ | ~~$B$~~ | $C$ |

| $C$ | $J$ | | |
|---|---|---|---|
| $A$ | $2$ | $1$ | $3$ |
| $B$ | $1$ | $3$ | $2$ |
| $C$ | $1$ | $2$ | $3$ |

| $C$ | $Day\ 1$ | $Day\ 2$ | $Day\ 3$ | $Day\ 4$ | |
|---|---|---|---|---|---|
| $A$ | 1,~~3~~ | 1 | ~~1~~, 2 | 2 | |
| $B$ | 2 | ~~2~~, 3 | 3 | 1,~~3~~ | |
| $C$ | | | | | |

Step 3: Job update preference

# Problem 1

| $J$ | $C$ | | |
|---|---|---|---|
| $1$ | ~~$A$~~ | $B$ | $C$ |
| $2$ | ~~$B$~~ | $A$ | $C$ |
| $3$ | ~~$A$~~ | ~~$B$~~ | $C$ |

| $C$ | $J$ | | |
|---|---|---|---|
| $A$ | $2$ | $1$ | $3$ |
| $B$ | $1$ | $3$ | $2$ |
| $C$ | $1$ | $2$ | $3$ |

| $C$ | $Day\ 1$ | $Day\ 2$ | $Day\ 3$ | $Day\ 4$ | $Day\ 5$ |
|---|---|---|---|---|---|
| $A$ | 1, ~~3~~ | 1 | ~~1~~, 2 | 2 | |
| $B$ | 2 | ~~2~~, 3 | 3 | 1, ~~3~~ | |
| $C$ | | | | | |

# Problem 1

| $J$ | $C$ | | |
|---|---|---|---|
| $1$ | ~~$A$~~ | $B$ | $C$ |
| $2$ | ~~$B$~~ | $A$ | $C$ |
| $3$ | ~~$A$~~ | ~~$B$~~ | $C$ |

| $C$ | $J$ | | |
|---|---|---|---|
| $A$ | $2$ | $1$ | $3$ |
| $B$ | $1$ | $3$ | $2$ |
| $C$ | $1$ | $2$ | $3$ |

| $C$ | $Day\ 1$ | $Day\ 2$ | $Day\ 3$ | $Day\ 4$ | $Day\ 5$ |
|---|---|---|---|---|---|
| $A$ | 1, ~~3~~ | 1 | ~~1~~, 2 | 2 | 2 |
| $B$ | 2 | ~~2~~, 3 | 3 | 1, ~~3~~ | 1 |
| $C$ | | | | | 3 |

# Problem 1

**Obs 1:** After each day we cross out at least one entry

- Guarantee terminates $\leq n^2$ days
- Tighter bound $(n-1)^2 + 1$ days

| $J$ | $C$ | | |
|---|---|---|---|
| $1$ | ~~$A$~~ | $B$ | $C$ |
| $2$ | ~~$B$~~ | $A$ | $C$ |
| $3$ | ~~$A$~~ | ~~$B$~~ | $C$ |

| $C$ | $J$ | | |
|---|---|---|---|
| $A$ | $2$ | $1$ | $3$ |
| $B$ | $1$ | $3$ | $2$ |
| $C$ | $1$ | $2$ | $3$ |

**Obs 2:** Candidates always becomes happier

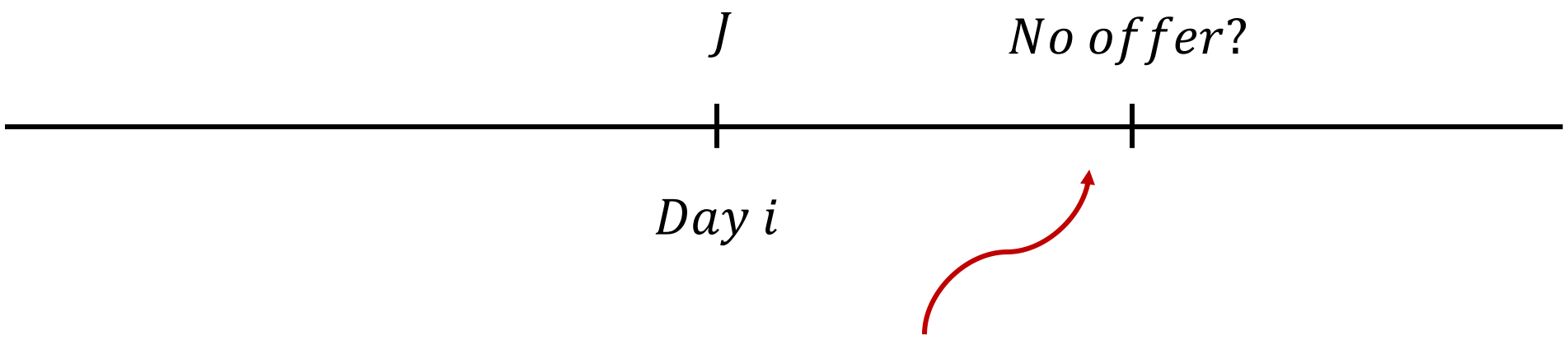
- Candidate takes ”max”

**Obs 3:** Jobs always becomes worse off.

| $C$ | $Day\ 1$ | $Day\ 2$ | $Day\ 3$ | $Day\ 4$ | $Day\ 5$ |
|---|---|---|---|---|---|
| $A$ | 1,~~3~~ | 1 | ~~1~~, 2 | 2 | 2 |
| $B$ | 2 | ~~2~~, 3 | 3 | 1,~~3~~ | 1 |
| $C$ | | | | | 3 |

# Problem 2 Propose-and-Reject Proofs

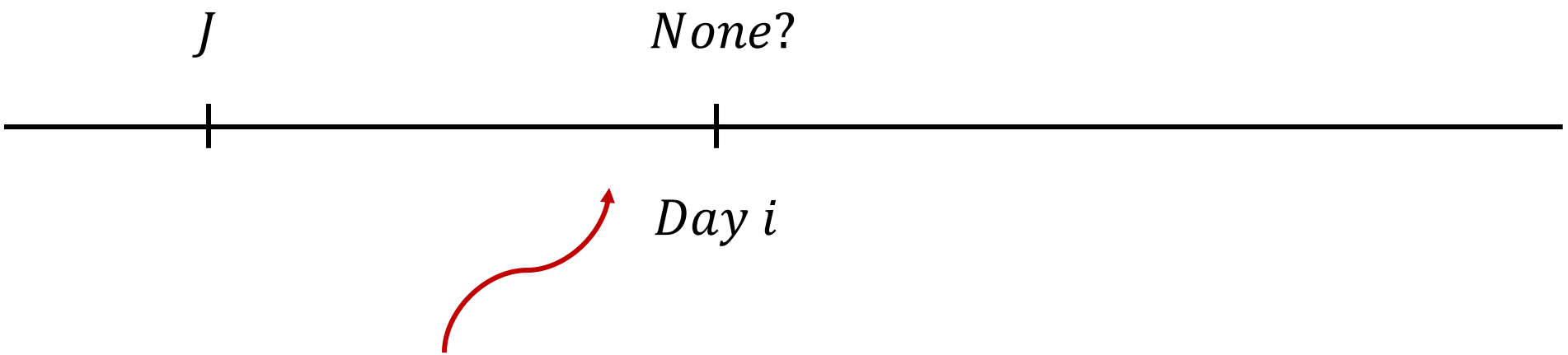
(a) In any execution of the algorithm, if a candidate receives a proposal on day $i$, then they receive some proposal on every day thereafter until termination




Candidates always gets better off over time (violates improvement lemma)!

# Problem 2 Propose-and-Reject Proofs

(b) In any execution of the algorithm, if a candidate receives no proposal on day $i$, then they receive no proposal on any previous days




Candidates always gets better off over time (violates improvement lemma)!

# Problem 2 Propose-and-Reject Proofs

(c) In any execution of the algorithm, there is at least one candidate who only receives a single proposal

Suppose the algorithm takes $k$ days.

$\Longrightarrow$ The algorithm does not terminate on day $k-1$

$\Longrightarrow$ At least one candidate $C$ does not receive an offer on day $k-1$

$\Longrightarrow$ $C$ does not receive any offer prior to day $k-1$

$\Longrightarrow$ $C$ receives its first (and only) proposal on day $k$

# Problem 3 Be a Judge

(a) There is a stable matching instance for $n$ jobs and $n$ candidates, such that in a stable matching algorithm with jobs proposing, every job ends up with its least preferred candidate.

| $J$ | $C$ | | |
|---|---|---|---|
| 1 | ✗ | ✗ | $A$ |
| 2 | ✗ | ✗ | $B$ |
| 3 | ✗ | ✗ | $C$ |

$\Longrightarrow$ Each job proposed to every candidate

$\Longrightarrow$ Each candidate recieves a proposal from every job

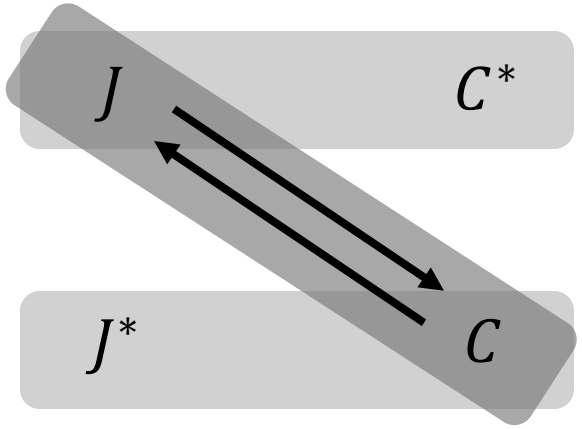
But by 2c, there is at least one candidate who recieves a single proposal!

**False**

# Problem 3 Be a Judge

(b) In a stable matching instance, if job $J$ and candidate $C$ each put each other at the top of their respective preference lists, then $J$ must be paired with $C$ in every stable pairing.

What if not?




- $J$ prefers $C$ over $C^*$
- $C$ prefers $J$ over $J^*$

**True**

# Problem 3 Be a Judge

(c) In a stable matching instance with at least two jobs and two candidates, if job $J$ and candidate $C$ each put each other at the bottom of their respective preference lists, then $J$ cannot be paired with $C$ in any stable pairing.

No idea whether to prove or disprove…

Try small cases

| $J$ | $C$ | |
|---|---|---|
| $1$ | $A$ | $B$ |
| $2$ | $A$ | $B$ |

| $C$ | $J$ | |
|---|---|---|
| $A$ | $1$ | $2$ |
| $B$ | $1$ | $2$ |

$\{(1, A), (2, B)\}$ is a stable pairing!

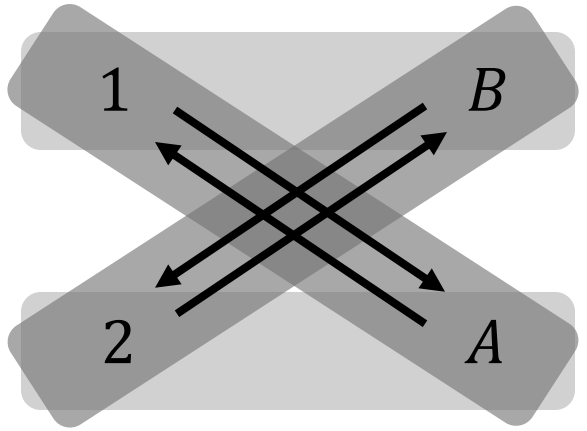
**False**

# Problem 3 Be a Judge

(d) For every $n > 1$, there is a stable matching instance for $n$ jobs and $n$ candidates which has an unstable pairing where every unmatched job-candidate pair is a rogue couple or pairing.

| $J$ | $C$ | |
|---|---|---|
| *1* | $A$ | $B$ |
| *2* | $B$ | $A$ |

| $C$ | $J$ | |
|---|---|---|
| $A$ | *1* | *2* |
| $B$ | *2* | *1* |




**True**