Induction

CS 70 Discussion 1A

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Note: These slides are unofficial course materials. Please use the notes as the only single source of truth.

Problem 1: Natural Induction on Inequality

Prove that if $n \in \mathbb{N}$, $x > 0$, then $(1 + x)^n \geq 1 + nx$.

Note:

- Base case: Prove the statement for $n = 0$.
- Induction hypothesis: Assume $n = k$ is true.
- Induction step: Prove the statement for $n = k + 1$

If the statement is true for $n = k$,
Then the statement is true for $n = k + 1$

$P(0)$	$P(1)$	$P(2)$	$P(3)$	……
Base case	Since $P(0)$ is true $P(1)$ is true	Since $P(1)$ is true $P(2)$ is true	Since $P(2)$ is true $P(3)$ is true	

Problem 1: Natural Induction on Inequality

Prove that if $n \in \mathbb{N}$, $x > 0$, then $(1+x)^n \geq 1 + nx$

Base case: $n = 0$

Check

$$(1+x)^0 = 1 \geq 1 + 0 \cdot x$$

The case $n = 1$ and $n = 2$ is also easy

$$(1+x)^1 = 1 + x \geq 1 + 1x$$

$$(1+x)^2 = 1 + 2x + x^2 \geq 1 + 2x$$

Problem 1: Natural Induction on Inequality

Prove that if $n \in \mathbb{N}$, $x > 0$, then $(1 + x)^n \geq 1 + nx$

What about $n = 3$?

How should we connect previous cases ($n = 2$) to current case ($n = 3$)?

$$(1 + x)^3 = \underline{(1 + x)^2}(1 + x)$$
$$\geq \underline{(1 + 2x)}(1 + x) = 1 + 3x + x^2$$
$$\geq 1 + 3x$$

Always nonnegative!

Now we've figured out how to jump from $n = k$ to $n = k + 1$!

Problem 1: Natural Induction on Inequality

Prove that if $n \in \mathbb{N}$, $x > 0$, then $(1+x)^n \geq 1 + nx$

Inductive hypothesis: Assume $n = k$ is true, i.e. $(1+x)^k \geq 1 + kx$ for some arbitrary k

Inductive step: Want to prove that $(1+x)^{k+1} \geq 1 + (k+1)x$

$$
\begin{aligned}
(1+x)^{k+1} &= (1+x)^k(1+x) \\
&\geq (1+kx)(1+x) = 1 + (k+1)x + x^2 \\
&\geq 1 + (k+1)x
\end{aligned}
$$

General strategy:

- Get some intuition by asking how can you go from $n = 1$ to $n = 2$, etc…
- Write down all steps to get partial credits

Problem 2: Make it Stronger

Base case: $n = 1$

$$a_1 = 1 \leq 3^{2^1} = 9$$

Inductive hypothesis: Suppose $n = k$ is true, i.e. $a_k \leq 3^{2^k}$

Inductive step: Prove that $n = k + 1$ is true , i.e . $a_{k+1} \leq 3^{2^{k+1}}$

How can we go from $n = k$ to $n = k + 1$?

$$a_{k+1} = 3a_k^2$$

$$\leq 3\left(3^{2^k}\right)^2 = \underline{3}(3^{2^{k+1}})$$

$$\nleq 3^{2^{k+1}}$$

There's an extra factor of 3!

Problem 2: Make it Stronger

Let’s instead prove that $a_n \leq \frac{1}{3} \cdot 3^{2^n} = 3^{2^n - 1}$

Base case: $n = 1$

$$a_1 = 1 \leq 3^{2^1 - 1} = 3$$

Inductive hypothesis: Suppose $n = k$ is true, i.e. $a_k \leq 3^{2^k - 1}$

Inductive step: Prove that $n = k + 1$ is true, i.e. $a_{k+1} \leq 3^{2^{k+1} - 1}$

$$\begin{aligned} a_{k+1} &= 3a_k^2 \\ &\leq 3\left(3^{2^k - 1}\right)^2 = 3(3^{2^{k+1} - 2}) \\ &= 3^{2^{k+1} - 1} \end{aligned}$$

Problem 3: Binary Numbers

Converting binary number to decimal

1	0	0	1	1	0	1
2^6	2^5	2^4	2^3	2^2	2^1	2^0

$$2^6 + 2^3 + 2^2 + 2^0 = 77$$

Any binary number $c_n c_{n-1} \dots c_1 c_0$ can be represented in decimal as

$$c_n 2^n + c_{n-1} 2^{n-1} + \cdots + c_1 2^1 + c_0 2^0$$

Binary Numbers ⇄ Decimal Numbers

Prove using induction!

Problem 3: Binary Numbers

Let's try induction

Base case: $n = 0$ has binary representation of 0

Inductive hypothese: Suppose $n = k$ has some binary representation

Inductive step: Prove that $n = k + 1$ is true.

- If k is even

$$\Longrightarrow k = c_m 2^m + c_{m-1} 2^{m-1} + \cdots + c_1 2^1$$

By induction hypothesis,
k has some binary representation,

$$\Longrightarrow k + 1 = c_m 2^m + c_{m-1} 2^{m-1} + \cdots + c_1 2^1 + 1$$

- If k is odd?

Problem 3: Binary Numbers

Let's try induction

Base case: $n = 0$ has binary representation of 0

Inductive hypothese: Suppose $n = k$ has some binary representation

Inductive step: Prove that $n = k + 1$ is true.

- If k is odd?

$\Longrightarrow k = c_m 2^m + c_{m-1} 2^{m-1} + \cdots + c_1 2^1 + 1$

$\Longrightarrow k + 1 = c_m 2^m + c_{m-1} 2^{m-1} + \cdots + c_1 2^1 + 2$

$\Longrightarrow k + 1 = c_m 2^m + c_{m-1} 2^{m-1} + \cdots + (c_1{+}1) 2^1$

$\Longrightarrow \cdots$

Problem 3: Binary Numbers

Let's try induction

Base case: $n = 0$ has binary representation of 0

Inductive hypothesis: Suppose $n \leq k$ has some binary representation

Inductive step: Prove that $n = k + 1$ is true.

- If k is odd?

$\Longrightarrow k + 1$ is even $\Longrightarrow {}^{k+1}/_{2}$ is an integer that is $\leq k$

$\Longrightarrow$ It has a binary representation $\Longrightarrow \frac{k+1}{2} = c_m 2^m + c_{m-1} 2^{m-1} + \cdots + c_1 2^1 + c_0$

$\Longrightarrow k + 1 = c_m 2^{m+1} + c_{m-1} 2^m + \cdots + c_1 2^2 + c_0 2^1$