Probability Theory

CS 70 Discussion 10A

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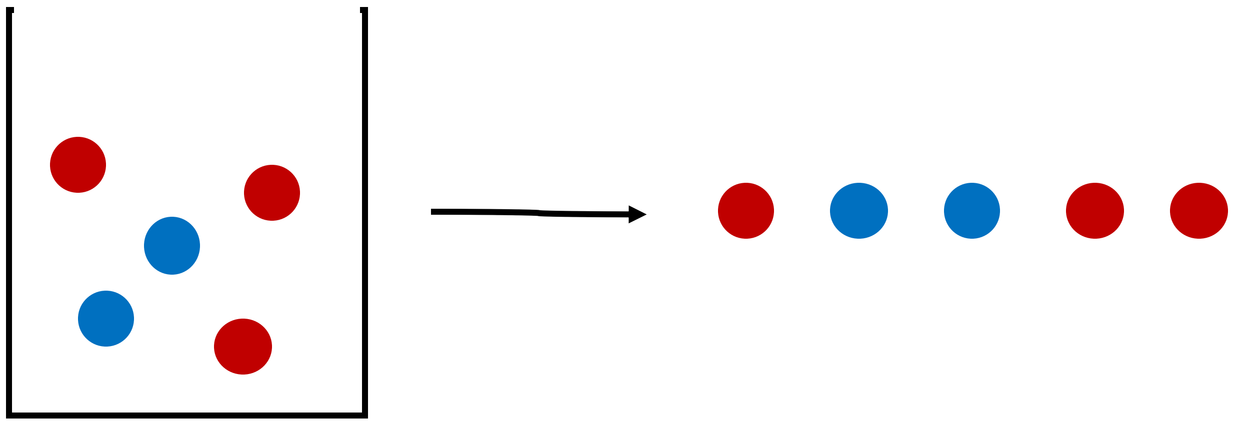
2025-04-02

Problem 1: Symmetry

Suppose there are $m = 3$ red marbles and 2 blue marbles

Experiment: Sample all 5 balls one by one, what is the sample space?



One possible outcome of the experiment

Problem 1: Symmetry

Sample space:

Probability function?

Way 2:

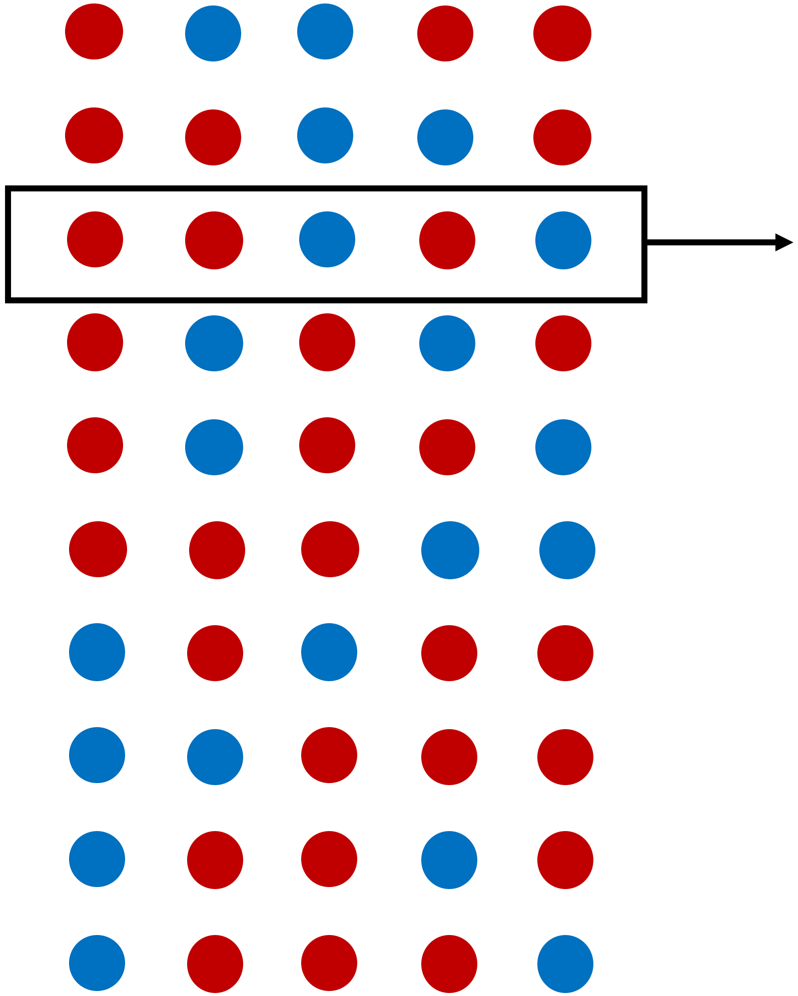
How many ways to arrange 3 red balls and 2 blue balls?

$$|\Omega| = \binom{5}{2}$$

___ ___ ___ ___ ___

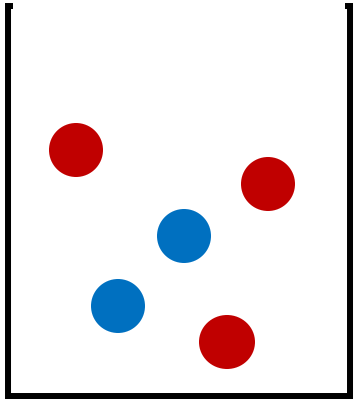
1 2 3 4 5

Sample 3 index from $\{1, 2, 3, 4, 5\}$ and place red.

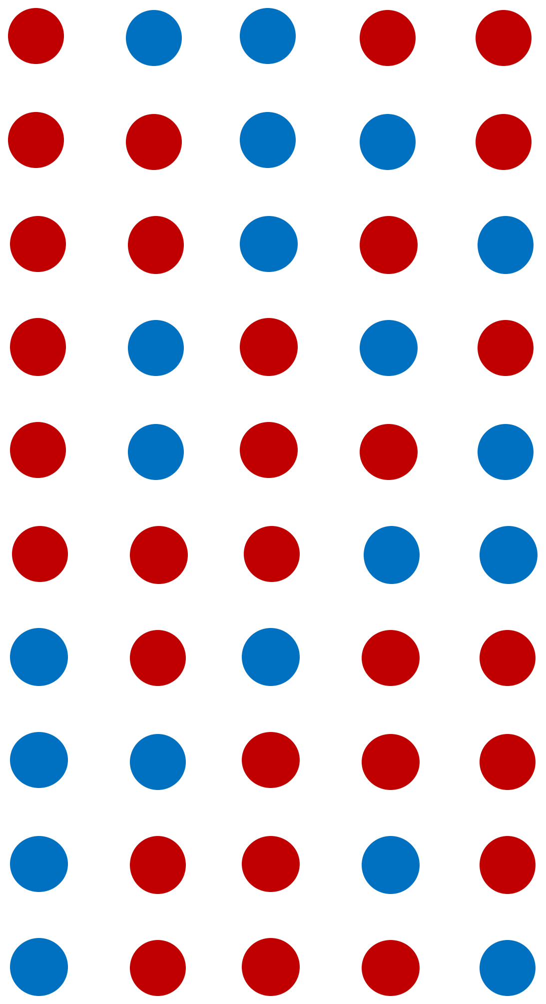


Way 1:

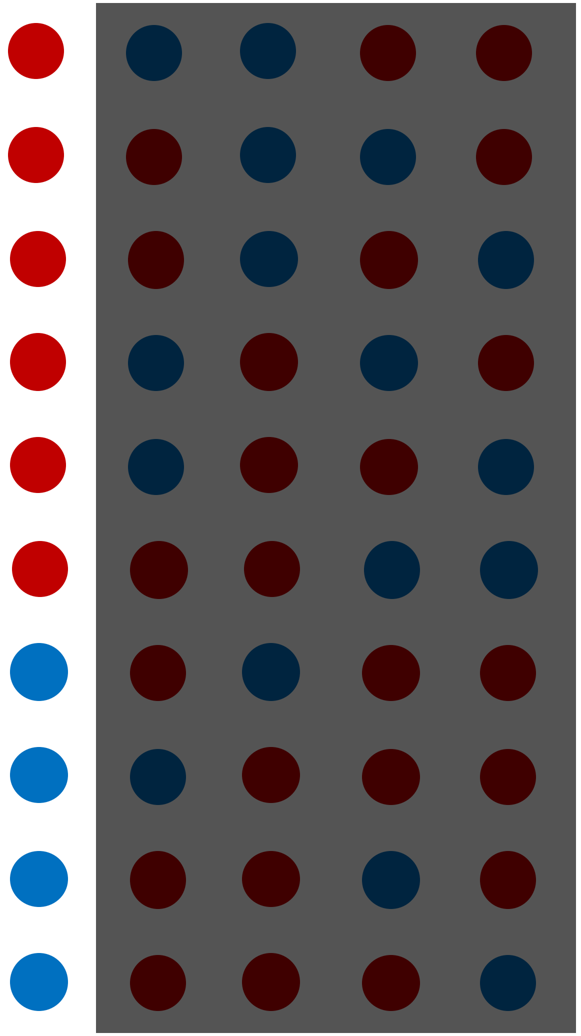
$$\frac{3}{5} \cdot \frac{2}{4} \cdot \frac{2}{3} \cdot \frac{1}{2} \cdot \frac{1}{1} = \frac{1}{10}$$



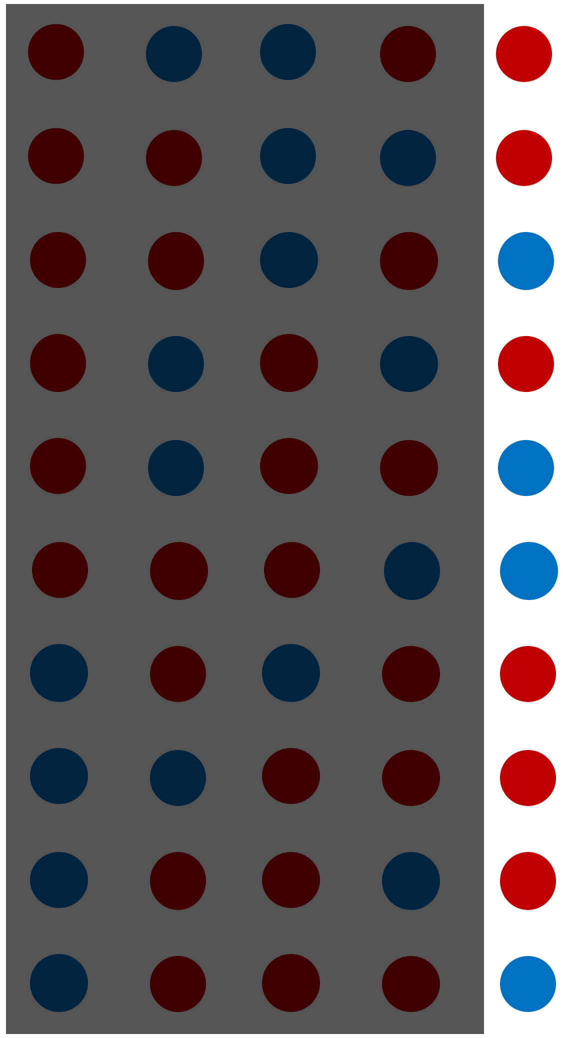
Problem 1: Symmetry



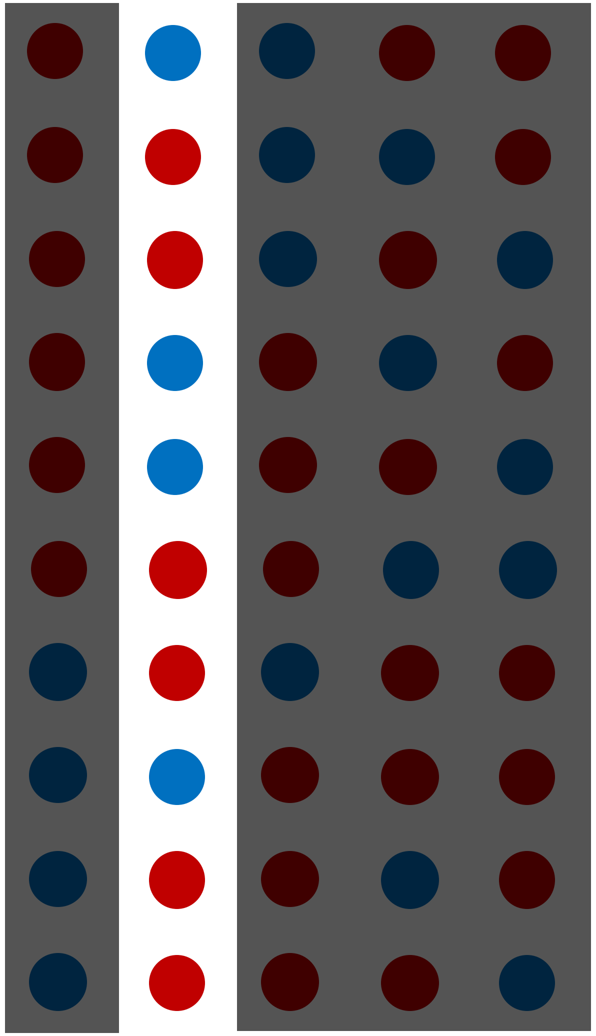
Problem 1: Symmetry



Problem 1: Symmetry



Problem 1: Symmetry

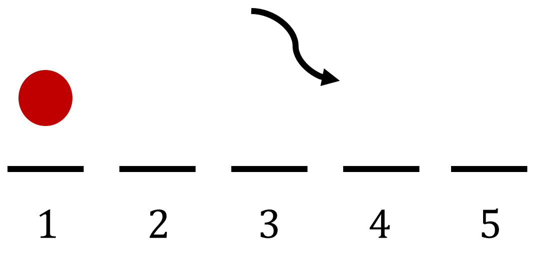


Problem 1: Symmetry

What is the probability that the first balls is red, without looking at the color of the other balls?

$$\mathbb{P}[A] = \frac{|A|}{\binom{n}{m}} = \frac{\binom{n-1}{m-1}}{\binom{n}{m}} = \frac{m}{n}$$

How many ways to arrange 2 red balls and 2 blue balls.



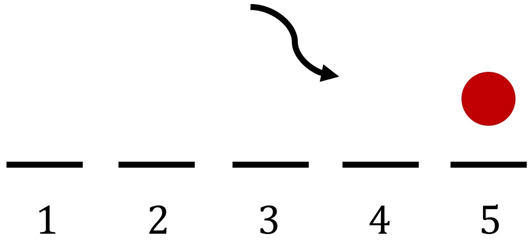
A: $\binom{4}{2}$ ways!

Problem 1: Symmetry

What is the probability that the final ball is red without looking at the color of the other balls?

$$\mathbb{P}[\mathrm{A}] = \frac{|\mathrm{A}|}{\binom{n}{m}} = \frac{\binom{n-1}{m-1}}{\binom{n}{m}} = \frac{m}{n}$$

How many ways to arrange 2 red balls and 2 blue balls.



A: $\binom{4}{2}$ ways!

Problem 1: Symmetry

Probability that the first ball is red without looking at the colors of the other balls A_1

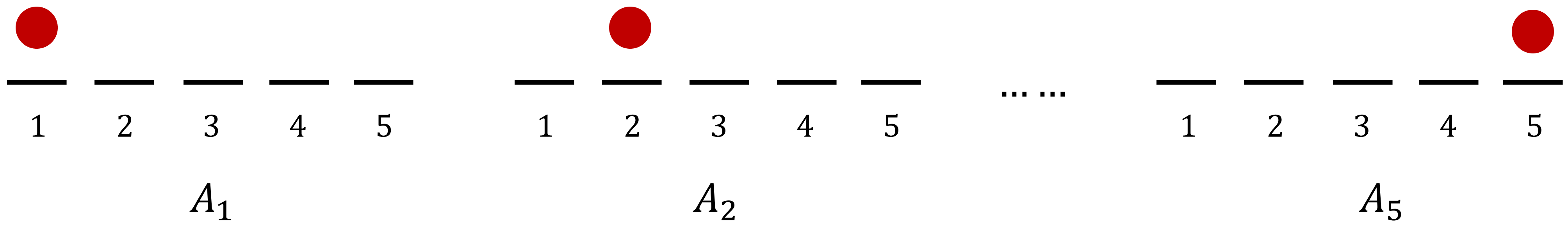
$\parallel$

Probability that the second ball is red without looking at the colors of the other balls A_2

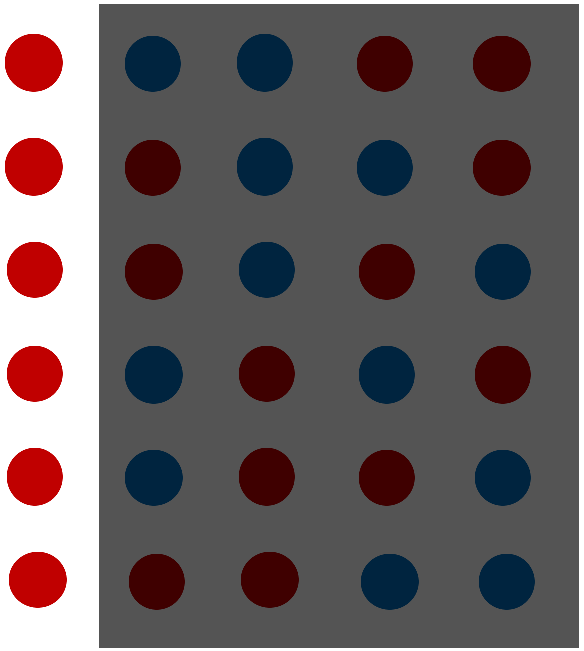
$\parallel$

$\vdots$

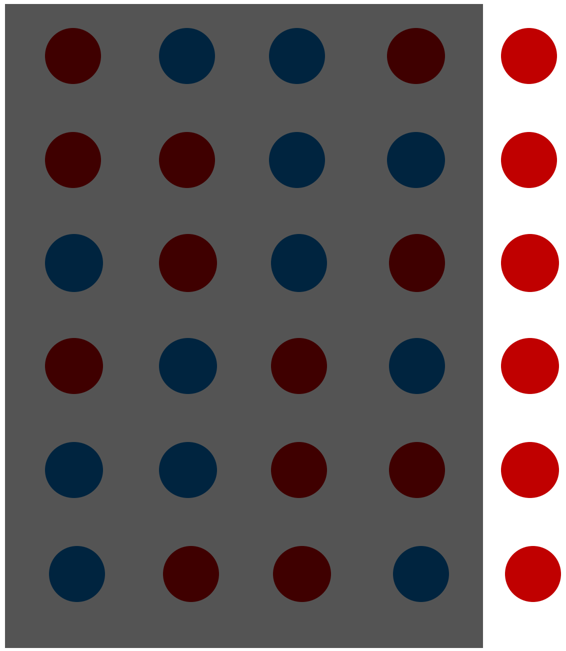
Probability that the final ball is red without looking at the colors of the other balls A_n



Problem 1: Symmetry

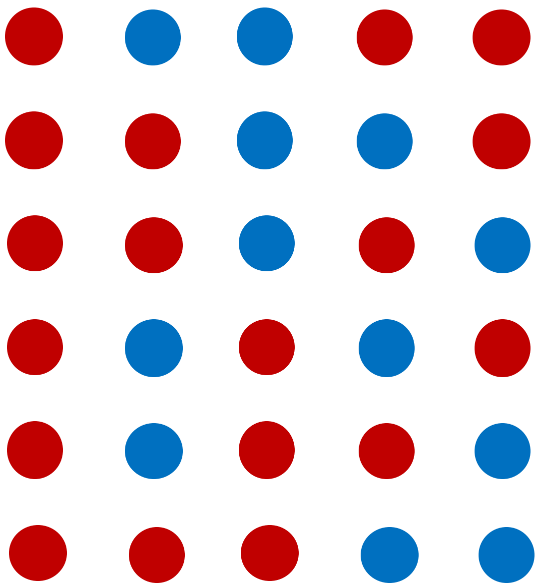


A_1

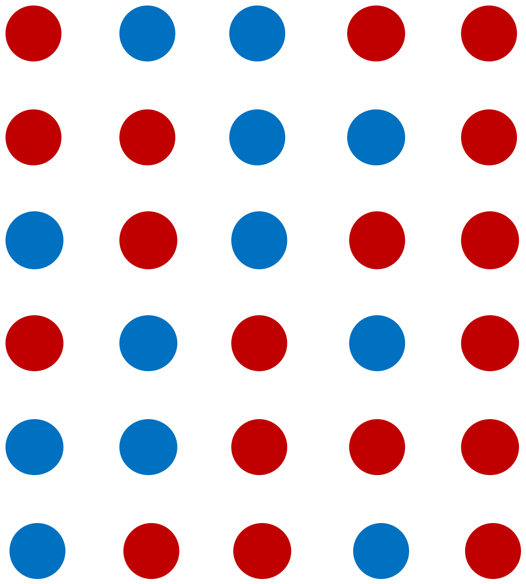


A_5

Problem 1: Symmetry

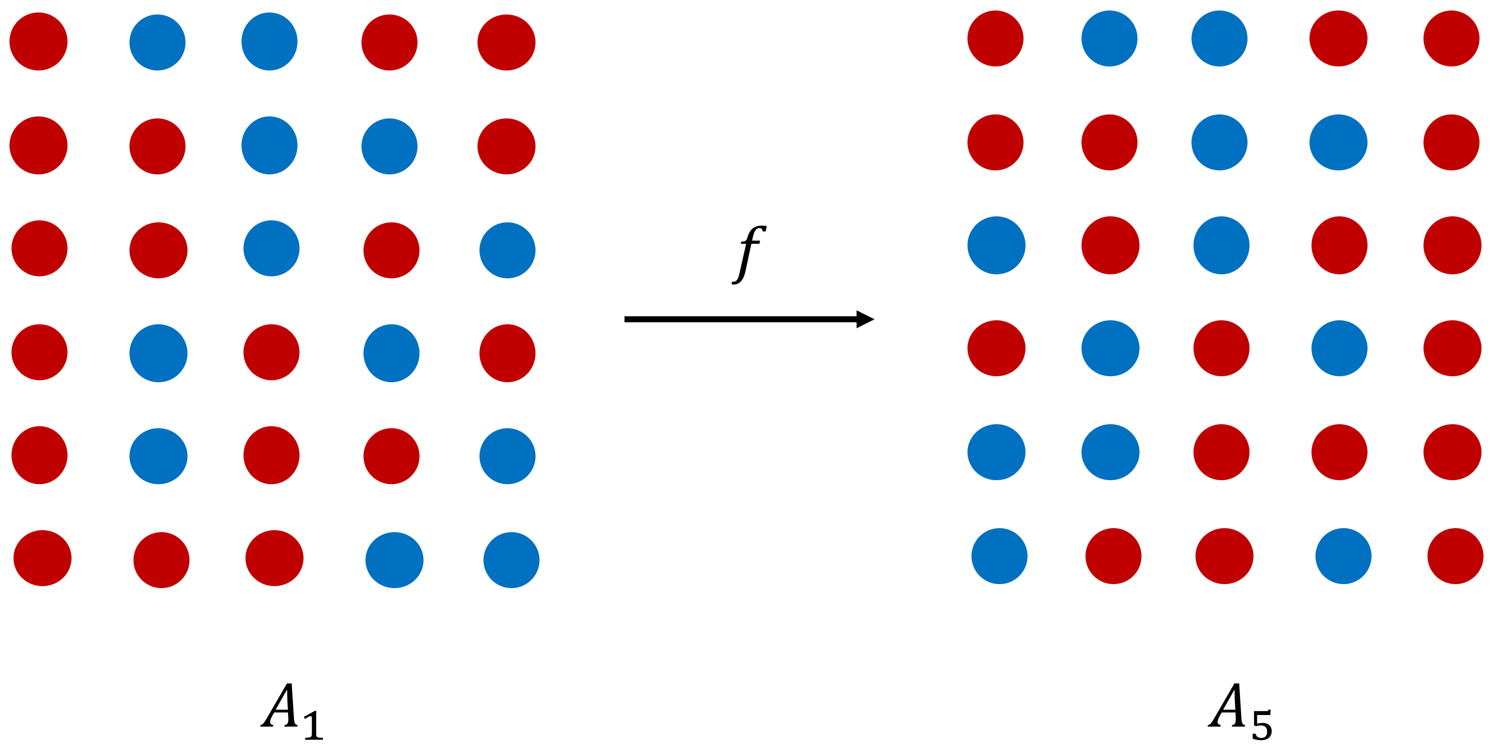


A_1



A_5

Problem 1: Symmetry



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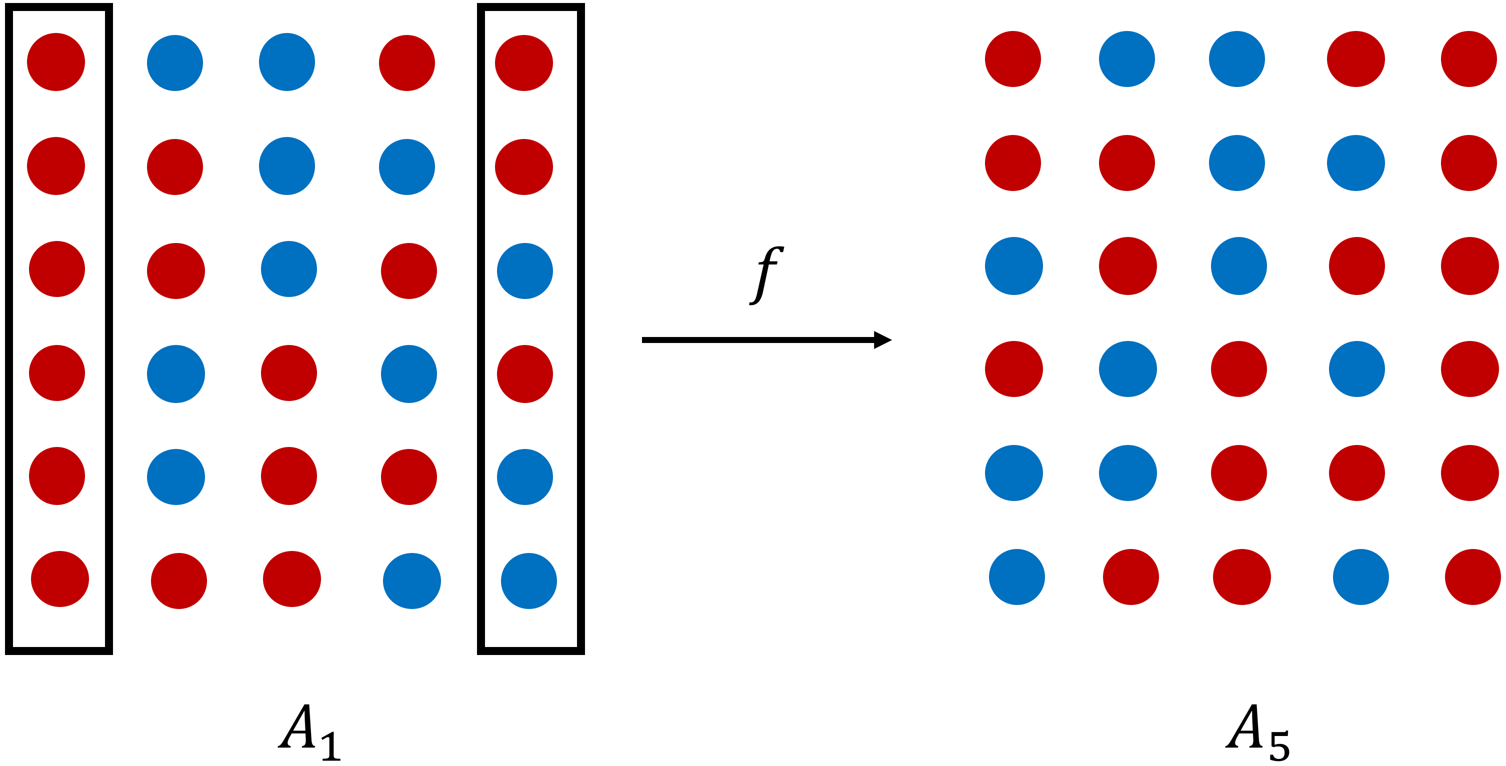

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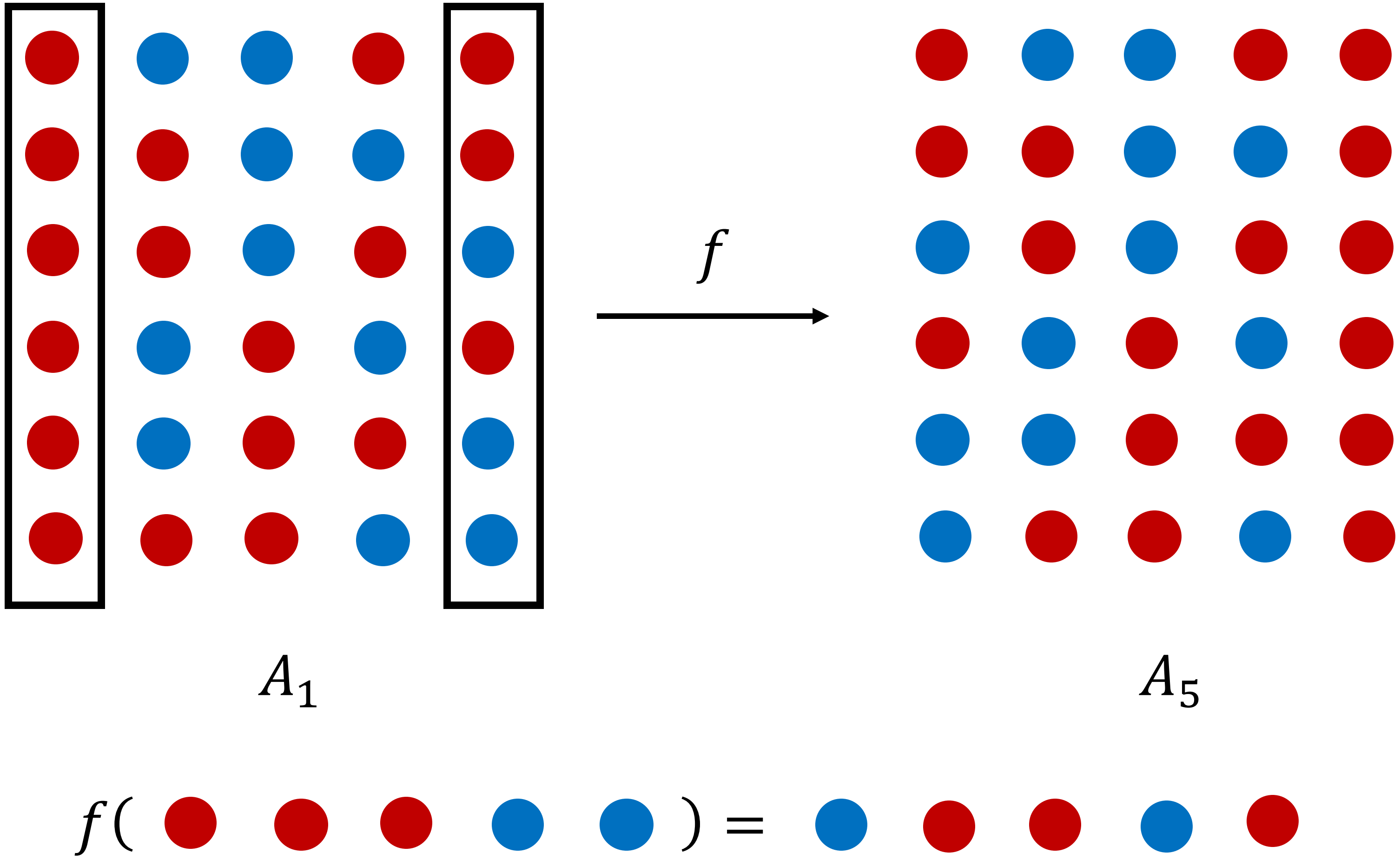
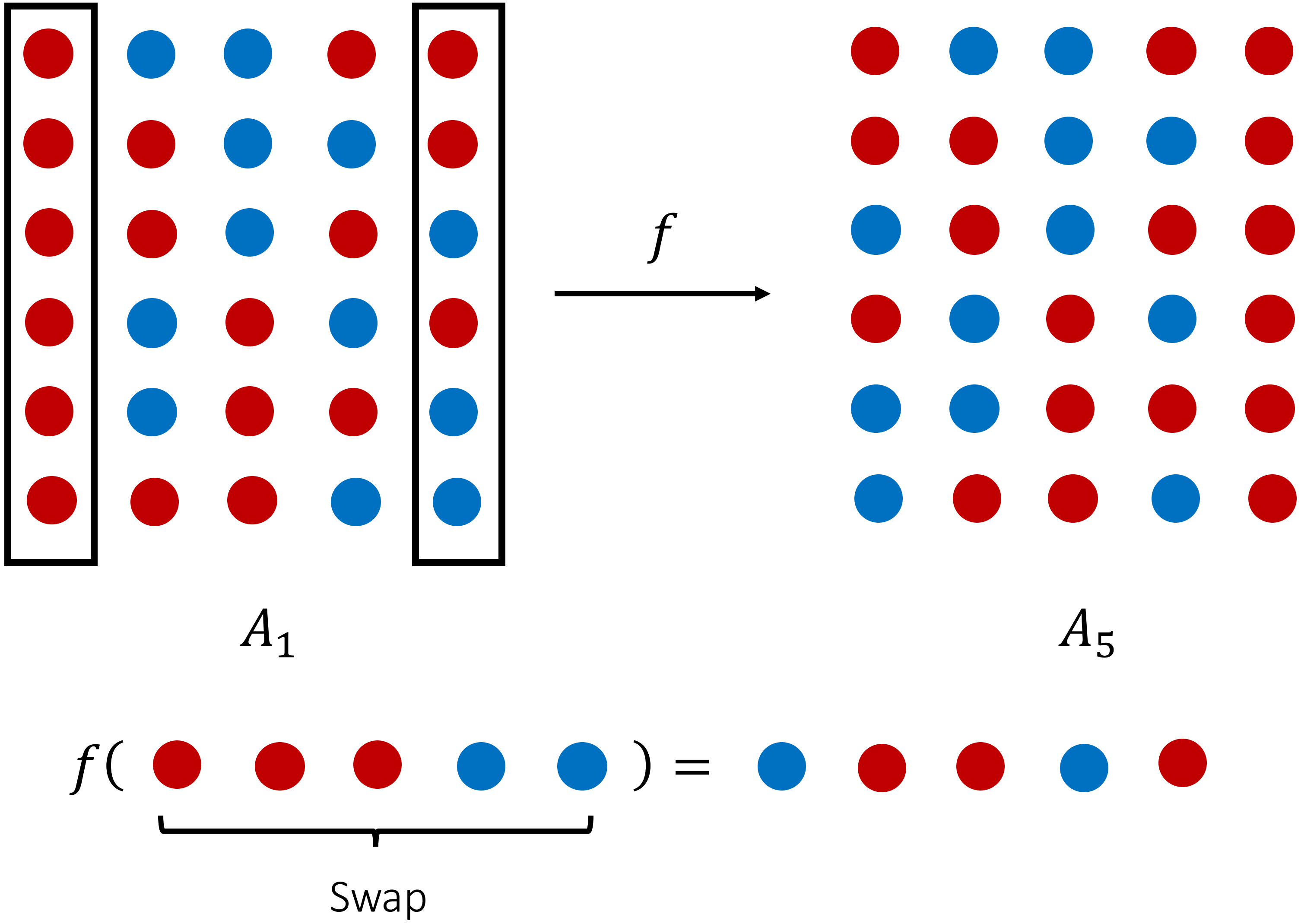
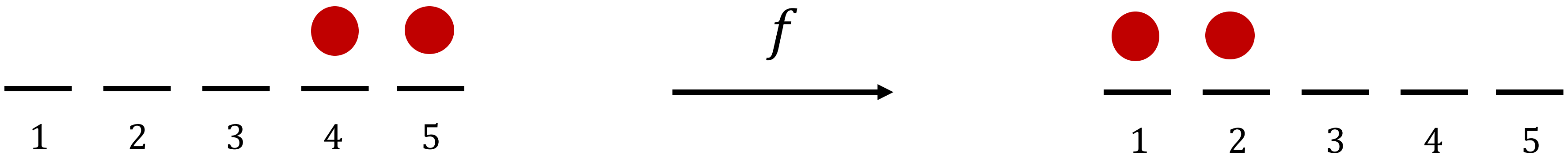


Problem 1: Symmetry



Problem 1: Symmetry

What is the probability that the last two ball is red without looking at the color of the other balls?



Swap column 4, 5 with column 1, 2

$$\mathbb{P}[\mathrm{A}] = \frac{\mathrm{m}}{\mathrm{n}} \cdot \frac{m-1}{n-1}$$

Problem 1: Symmetry

What is the probability that the second-to-last ball is red given that the first ball is red?

A B

$$\mathbb{P}[A|B] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]}$$

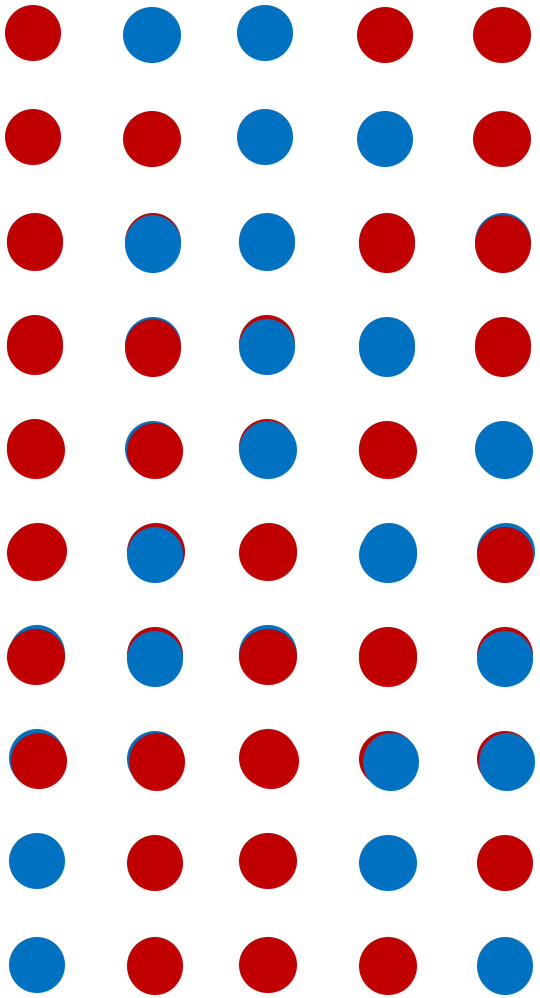
$$= \frac{\frac{m}{n} \cdot \frac{m-1}{n-1}}{\frac{m}{n}}$$

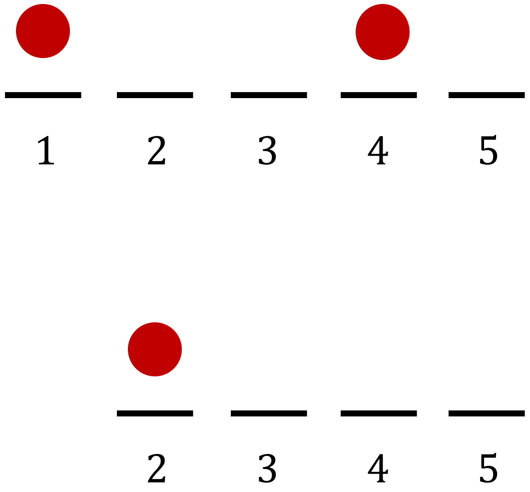
$$= \frac{m-1}{n-1}$$

the first and second-to-last ball is red:

Problem 1: Symmetry

$$\frac{1}{\binom{n-1}{m-1}} \longrightarrow$$





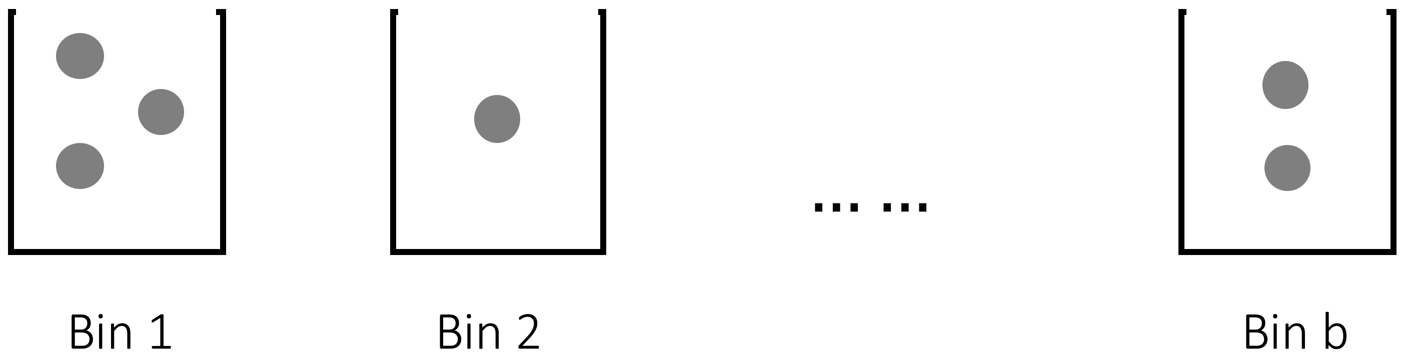
$$\uparrow$$

$$\frac{m-1}{n-1}$$

Problem 2: Balls and Bins

Suppose you throw b balls into n bins one at a time

What is the sample space?

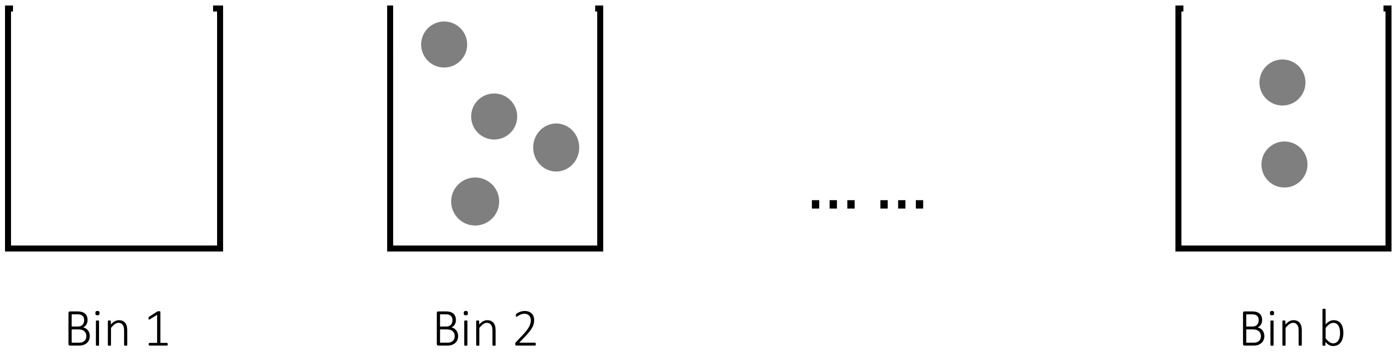


What is the probability function?

$$\mathbb{P}[w] = \frac{1}{n} \cdot \frac{1}{n} \cdot \ldots \cdot \frac{1}{n} = \frac{1}{n^b}$$

Problem 2: Balls and Bins

What is the probability first bin is empty?



$$\mathbb{P}[w] = \frac{(n-1)}{n} \cdot \frac{(n-1)}{n} \cdot \ldots \cdot \frac{(n-1)}{n} = \frac{(n-1)^b}{n^b}$$

Ball 1 cannot fall into bin 1

Ball 2 cannot fall into bin 1

Problem 2: Balls and Bins

What is the probability first k bin is empty?

$$\mathbb{P}[w] = \frac{(n-k)}{n} \cdot \frac{(n-k)}{n} \cdot \ldots \cdot \frac{(n-k)}{n} = \frac{(n-k)^b}{n^b}$$

Ball 1 cannot fall into bin 1,..,k

Ball 2 cannot fall into bin 1,..k

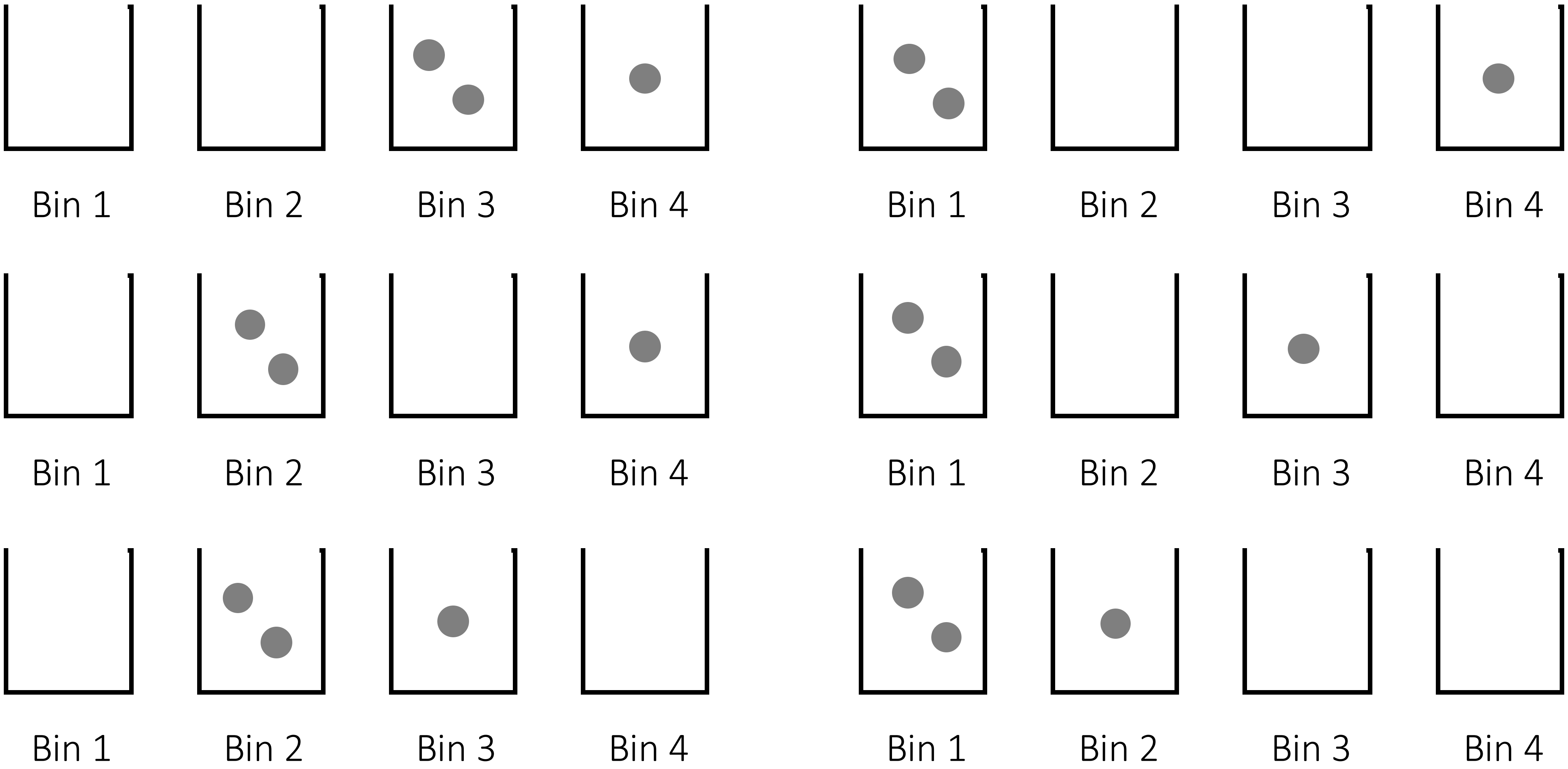
What is the probability at least k bin is empty?

Trick:

$$\mathrm{A} = \bigcup_{i=1}^{m} A_i$$

ith set of k bins is empty

Problem 2: Balls and Bins



Problem 2: Balls and Bins

What is the probability at least k bin is empty?

$$\mathbb{P}\left[\bigcup_{i=1}^{m} A_i\right] \leq \sum_{i=1}^{m} \mathbb{P}[A_i]$$

$$= \sum_{i=1}^{m} \frac{(n-k)^b}{n^b}$$

$$= \binom{n}{k} \cdot \frac{(n-k)^b}{n^b}$$