Proofs

CS 70 Discussion 0B

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Welcome to CS70!

A few tips

- Scan through the notes before going to the lecture and discussion.
- It takes time to digest the material, so don’t procrastinate.
- Try to build intuition.
- Ask questions.

Problem 1a: Propositional Logic Review

Suppose P is true, Q is false.

- $P \land Q$ (P and Q): False
- $P \lor Q$ (P or Q): True
- $P \implies Q$ (P implies Q): What???

P	Q	$P \Rightarrow Q$
F	T	T
F	F	T
T	T	T
T	F	F

How to evaluate the implication $P \implies Q$?

Logically equivalent to $\neg P \lor Q$!

- $P \implies Q \equiv \neg P \lor Q$: False
- $\neg P \implies Q \equiv P \lor Q$: True
- $\neg Q \implies \neg P \equiv Q \lor \neg P$: False

$P \implies Q$ is the same as $\neg Q \implies \neg P$!

Problem 1b: Propositional Logic Review

Quantifiers allow us to create more complex propositions

- $\forall x$ (for all x)
- $\exists x$ (exists x)

Every integer has at least one divisor

for all $x \in \mathbb{Z}$ there exist a divisor y

for all $x \in \mathbb{Z}$ there exist y s.t. y divides x

$$(\forall x \in \mathbb{Z})(\exists y \in \mathbb{Z})\ (y \text{ divides } x)$$

Problem 1c: Propositional Logic Review

$$\underline{\neg}\,\underline{(\exists x \in \mathbb{Z})}\,\underline{(\forall y \in \mathbb{Z})}\,\underline{(x > y)}$$

No	there exist integer x	for all integer y	$x > y$
It is not the case that	there exist integer x such that	for all integer y	$x > y$

There is no largest integer

General strategy:

- Read the statement by small parts
- Sometimes need to insert “such that” to make sense
- Try to make the final statement as concise as possible

Problem 3: Perfect Square

(a) Prove that if n^2 is odd, then n is odd.

(b) Prove that if n^2 is odd, then $n^2 = 8k + 1$ for some k.

Note:

Suppose you want to prove $P \implies Q$ (if P then Q)

- Direct proof: Assume P and try to show Q
- Proof by contrapositive: Assume $\neg Q$ and try to show $\neg P$
- Proof by contradiction: Assume both P and $\neg Q$ and try to derive a contradiction.

Write down the statement for each corresponding proof technique and try the one that seems easiest to you!

Problem 3a: Perfect Square

(a) Prove that if $\underbrace{n^2 \text{ is odd,}}_{P}$ then $\underbrace{n \text{ is odd.}}_{Q}$

Maybe direct proof?

Strategy: Assume $\underbrace{n^2 \text{ is odd,}}_{P}$ need to show $\underbrace{n \text{ is odd.}}_{Q}$

- n^2 is odd $\Longrightarrow n^2 = 2m + 1$ for some m
- Can we show that $n = 2k + 1$ for some k?
- Probably?

Problem 3a: Perfect Square

Proof by contradiction

$$\text{Strategy: Assume } \underbrace{n^2 \text{ is odd}}_{P} \text{ and } \underbrace{n \text{ is even}}_{\neg Q}\text{. Try to find a contradiction.}$$

- n is even $\Longrightarrow n = 2m$ for some m
- $\Longrightarrow n^2 = 4m^2$ is even, contradiction!

Proof by contraposition also works!

$$\text{Strategy: Assume } \underbrace{n \text{ is even}}_{\neg Q} \text{ and try to show that } \underbrace{n^2 \text{ is even}}_{\neg P}$$

Problem 3a: Perfect Square

Why does proof by contraposition and proof by contradiction work in this problem?

Note:

Suppose you want to prove $P \implies Q$ (if P then Q)

- **Direct proof:** Assume P and try to show Q
- **Proof by contraposition:** Assume $\neg Q$ and try to show $\neg P$
- **Proof by contradiction:** Assume both P and $\neg Q$ and try to derive a contradiction.

- Easy to go from n to n^2 but not the other way around.
- The negation of Q (i.e. n is even) is an easier assumption to work with!

Problem 3b: Perfect Square

Prove that if $\underbrace{n^2 \text{ is odd}}_{P}$, then $\underbrace{n^2 = 8k + 1 \text{ for some } k}_{Q}$.

- (Direct)

  Assume $\underbrace{n^2 \text{ is odd}}_{P}$ and try to show $\underbrace{n^2 = 8k + 1 \text{ for some } k}_{Q}$.

- (Contraposition)

  Assume $\underbrace{n^2 = 8k, 8k + 2{,}8k + 3, \ldots 8k + 7 \text{ for some } k}_{\neg Q}$ and try to show $\underbrace{n^2 \text{ is even}}_{\neg P}$

  Many cases… Maybe proof by case?

Problem 3b: Perfect Square

Try direct proof first:

Assume n^2 is odd and try to show $n^2 = 8k + 1$ for some k.

How to proceed?

CS70 Tip: Questions in series sometimes contain hints!

- n^2 is odd $\Longrightarrow$ n is odd (from part a) $\Longrightarrow n = 2m + 1$ for some m.
- $\Longrightarrow n^2 = 4m^2 + 4m + 1 = 4(\underline{m^2 + m}) + 1$
- $\Longrightarrow n^2 = 4m(m + 1) + 1$ ⟶
- Since $\mathrm{m}(m + 1)$ is even, it follows that $n^2 = 8k + 1$ for some k

If m odd, then $m + 1$ even

Otherwise m even

Problem 4: Numbers of Friends

Note (Pigeonhole principle)

Suppose $n > m$.

If there are n pigeons in m holes, then at least one hole contain more than one pigeon.

Problem 4: Numbers of Friends

If $\underbrace{n \geq 2 \text{ friends at a party}}_{P}$, then $\underbrace{\text{at least } 2 \text{ of them have the same number of friends}}_{Q}$

Analysis:

Note that $\neg Q$ is a strong statement: So maybe contrapositive or contradiction is the way to go?

None of them have the same number of friends

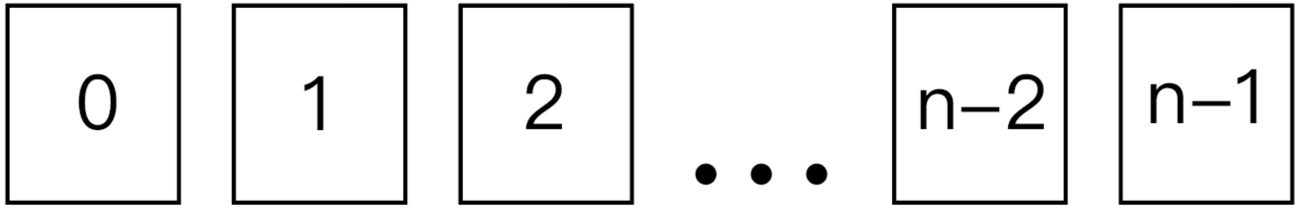
$\Longrightarrow$ The number of friends are distinct

$\Longrightarrow$ The number of friends is the set $\{0, 1, 2, 3, \dots n - 1\}$

What's wrong with this? Someone has 0 friends, yet someone has $n - 1$ friends. Contradiction!

Problem 4: Numbers of Friends

Where did we use pigeonhole principle?



- Bin 0 and Bin $n-1$ cannot be nonempty together
- So only $n-1$ bins, n pigeons (people)

Takeaways

Propositional logic

- $P \implies Q \equiv \neg P \vee Q$: great for proofs!
- Translate between words and logic: by small chunks

Proof:

- Write down the goals for each proof techniques in words!
- Is $\neg Q$ easier to work with? Try contrapositive or contradiction!
- Questions in series often contains hint
- Pigeonhole principle